THE DEVELOPMENT
OF A MATHEMATICAL MODEL
FOR PREDICTING SOLAR HEAT GAINS
THROUGH BUILDING WALLS AND ROOFS

by L. O. Degelman

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by

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Department of Architectural Engineering

The Pennsylvania State University
College of Engineering
Institute for Building Research

June 1966
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FOREWORD

The Better Building Report series was initiated to assist the builder of houses; later it was expanded to include the interests of the construction industry in general. Now the scope of this series is further expanded to report on research which, although not immediately applicable, should be of basic interest to the architectural profession.

The building designer, in particular, will be glad to learn that a means is in sight for taking the guesswork out of deciding which of various wall systems will provide the best performance at the most economical cost for a particular building. When the research discussed in this study is perfected, it will allow the designer of large building projects to take into consideration the heat gain into a building from solar radiation. Mr. Degelman's goal is the development of a computer language which will enable the designer to go to any computer center for this type of wall system evaluation. More recent research by the author has tested the mathematical models presented in this paper and has confirmed this method as accurate and dependable. Results of continued research in this field will constitute a later report in this series.

Another report which is expected to be published soon is: A Guide for Evaluating Ultimate Cost of Heating Systems. Future reports are expected to reflect such things as new ideas for builders, new combinations of materials, and the effects of air pollution on buildings.

E. R. Queer, Director
Institute for Building Research

University Park, Pennsylvania
June 27, 1966
ACKNOWLEDGMENTS

The author wishes to thank Professor G. H. Albright, Head of the Department of Architectural Engineering, for his help in the organization and background of this report; Professor John Everetts, Jr. and Professor E. R. Queer, Director of the Institute for Building Research at The Pennsylvania State University, for their expert advice in the area of air-conditioning operating cost analysis; Professor V. L. Pass for his helpful comments on the analysis of the "Air Wall Research Building" data; and Professor E. R. McLaughlin of the Institute for Building Research for his helpful comments and review of the entire report, especially in the area of heat flow calculations.

L. O. Degelman
Instructor of Architectural Engineering
BACKGROUND

The ability to be able to predict accurately the solar heat gain through walls and roofs would help a designer in selecting wall systems on a minimum total cost basis. This study, therefore, develops general mathematical models which will describe the heat gain into a building from the direct and diffuse solar radiation and indoor-outdoor temperature differentials. These models are intended for future use by a computer as a segment of the calculation of total air-conditioning loads on a building. In order to use such a mathematical model effectively, the author is in the process of developing a problem-oriented computer language.

The computer language will contain mathematical models for heat gain and heat loss calculations, a repertoire of building wall and roof systems, their thermal properties and costs, and a set of rules and regulations for the use of the language.

The requirements for using such a design language indicate a need for the availability of mathematical equations which will describe the sun's position, the periodic heat flow through walls, roof, and windows, and the total cooling and heating loads. Since the determination of the ultimate costs of wall and roof systems (including air-conditioning operating costs) is a requirement, the use of only the "design" conditions is not sufficient. To evaluate the costs properly, the total year-round operating costs are needed. Most of the available information on this subject is presented for design conditions only and not for year-round operating cost evaluation (Ref. 3). Therefore, the need has arisen for the development of mathematical equations which will accurately predict the operating costs of an air-conditioning system as well as determine the capacity requirement of the system. Using these models in a computer, a designer will be aided in the selection of wall systems using the ultimate cost (air-conditioning and maintenance costs plus initial costs) as a basis for his selection.

The research described in this study was done as part of Project MODCON (Man-Machine Optimum Design and Construction, see Refs. 1 and 2) within the Department of Architectural Engineering and the Institute for Building Research at The Pennsylvania State University. The research goal of Project MODCON is the development of a systematic approach to optimal building design and the production of construction documents for use in building construction. The techniques developed from this continuing project of research studies are intended for use by educational programs as well as by the building industry and the building design professions. For effective use of such techniques, the MODCON system integrates various phases of analog and digital computer applications and automation techniques in order to present decision-making information to the building designer and to communicate and document information for use by the building industry.
Areas of application to the building industry and construction process include the following:

- Computer-aided planning and design
- Building material and component selection
- Specifications development
- Building system design and performance calculations
- Cost analysis and estimates
- Construction scheduling
- Office management and scheduling

Automation techniques to accomplish the above applications would involve:

- Information storage and retrieval
- Visual image retrieval and manipulation
- Mathematical computation (digital and analog)
- Graphical communications through use of data plotters and cathode ray tubes

At present the MODCON project has progressed to the point of utilization of data plotters for drawing plans and elevations of buildings and the initiation and use of computer programs for information retrieval and subsystem design such as structural member design and solar control. Future efforts will be aimed at improving man-machine communications for building designers and developing mathematical models, such as the one contained in this report, for use in an overall system for design and evaluation of building systems.

### NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>φ</td>
<td>Latitude on earth (degree plus or minus indicating north or south latitudes respectively)</td>
</tr>
<tr>
<td>σ</td>
<td>Solar declination angle (degrees plus or minus)</td>
</tr>
<tr>
<td>θ</td>
<td>Sun's altitude angle (angle from horizon, degrees)</td>
</tr>
<tr>
<td>θ₂</td>
<td>Sun's zenith angle (complement of θ₁, degrees)</td>
</tr>
<tr>
<td>ψ</td>
<td>Sun's azimuth angle (measured from north, degrees)</td>
</tr>
<tr>
<td>ψ_R</td>
<td>Sun's azimuth angle at sunrise (degrees)</td>
</tr>
<tr>
<td>ψ_N</td>
<td>Azimuth angle to normal to wall surface (degrees)</td>
</tr>
<tr>
<td>θ₁</td>
<td>Angle between incident sun ray and normal to a surface (degrees)</td>
</tr>
<tr>
<td>d</td>
<td>Time of year (days from June 21)</td>
</tr>
<tr>
<td>T_R</td>
<td>Time of sunrise (hours from midnight)</td>
</tr>
<tr>
<td>T</td>
<td>Time of day (hours from midnight)</td>
</tr>
<tr>
<td>AST</td>
<td>Apparent solar time (hours)</td>
</tr>
<tr>
<td>MST</td>
<td>Mean solar time (hours)</td>
</tr>
<tr>
<td>ST</td>
<td>Standard time (hours)</td>
</tr>
<tr>
<td>ET</td>
<td>Equation of time (minutes)</td>
</tr>
</tbody>
</table>

### Incident Solar Energy and Atmospheric Conditions

- \( I_{DN} \) Normal direct solar radiation (Btu/hr ft\(^2\))
- \( I_{DH} \) Direct solar radiation on horizontal surface (Btu/hr ft\(^2\))
- \( I_{DV} \) Direct solar radiation on a vertical surface (Btu/hr ft\(^2\))
- \( I_{dh} \) Diffuse solar radiation on a horizontal surface (Btu/hr ft\(^2\))
- \( I_d \) Diffuse solar radiation on a vertical surface (Btu/hr ft\(^2\))
- \( I_r \) Total incident solar radiation (Btu/hr ft\(^2\))
- \( I_{sc} \) Ratio of solar radiation intensity outside the earth's atmosphere to the solar constant, dimensionless
- \( I_o \) Solar constant (442.4 Btu/hr ft\(^2\))
- \( I_{o} \) Apparent solar constant (Btu/hr ft\(^2\))
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>Atmospheric extinction coefficient, dimensionless</td>
</tr>
<tr>
<td>(\overline{H})</td>
<td>Monthly average daily total solar radiation on a horizontal surface at earth's surface (Btu/day, (\text{ft}^2))</td>
</tr>
<tr>
<td>(H_o)</td>
<td>Daily total solar radiation on a horizontal surface outside the earth's atmosphere (Btu/day, (\text{ft}^2))</td>
</tr>
<tr>
<td>(K_T)</td>
<td>Monthly average daily cloudiness index ((K_T = \overline{H}/H_o)), dimensionless</td>
</tr>
<tr>
<td>(H)</td>
<td>Daily total solar radiation on a horizontal surface at earth's surface (Btu/day, (\text{ft}^2))</td>
</tr>
<tr>
<td>(K_T)</td>
<td>Daily cloudiness index ((K_T = H/H_o)), dimensionless</td>
</tr>
<tr>
<td>(K_C)</td>
<td>Fraction of cloud cover, or one minus percent of possible sunshine, dimensionless</td>
</tr>
<tr>
<td>(D)</td>
<td>Daily diffuse solar radiation on a horizontal surface at earth's surface (Btu/day, (\text{ft}^2))</td>
</tr>
<tr>
<td>(K_d)</td>
<td>Ratio of daily diffuse solar radiation of earth's surface to total radiation on a horizontal surface outside the earth's atmosphere ((K_d = D/H_o)), dimensionless</td>
</tr>
<tr>
<td>(K_D)</td>
<td>Ratio of daily direct solar radiation on horizontal surface at earth's surface to the total radiation on a horizontal surface outside the earth's atmosphere ([K_D = (H - D)/H_o = K_T - K_d]), dimensionless</td>
</tr>
<tr>
<td>(B)</td>
<td>Ratio of instantaneous diffuse solar radiation on a horizontal surface to the instantaneous direct normal solar radiation ([B = I_{dn}/I_{DN}]), dimensionless</td>
</tr>
<tr>
<td>(r_d)</td>
<td>Transmission coefficient for diffuse solar radiation, or ratio of (I_{dn}) to extraterrestrial solar radiation on a horizontal surface, dimensionless</td>
</tr>
<tr>
<td>(r_D)</td>
<td>Transmission coefficient for direct solar radiation ([r_D = I_{DN}/I_{DN}]), dimensionless</td>
</tr>
</tbody>
</table>

**Heat Flow**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>Solar absorptivity of surface exposed to sun, dimensionless</td>
</tr>
<tr>
<td>(t_a)</td>
<td>Outdoor dry-bulb temperature of air (°F)</td>
</tr>
<tr>
<td>(t_e)</td>
<td>Sol-air temperature (°F)</td>
</tr>
<tr>
<td>(t_o)</td>
<td>Temperature of inside surface of wall (°F)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t_i)</td>
<td>Temperature of inside air (°F)</td>
</tr>
<tr>
<td>(h_o)</td>
<td>Indoor air film coefficient (Btu/hr, (\text{ft}^2), °F)</td>
</tr>
<tr>
<td>(h_L)</td>
<td>Outdoor air film coefficient (Btu/hr, (\text{ft}^2), °F)</td>
</tr>
<tr>
<td>(V_w)</td>
<td>Velocity of wind (mph)</td>
</tr>
<tr>
<td>(t_{M})</td>
<td>Daily average sol-air temperature (°F)</td>
</tr>
<tr>
<td>(\lambda_o)</td>
<td>Mean decrement factor (average of the 1st and 2nd harmonic), dimensionless</td>
</tr>
<tr>
<td>(\rho_c)</td>
<td>Volumetric specific heat (Btu/ft³, °F)</td>
</tr>
<tr>
<td>(L)</td>
<td>Thickness of wall (ft)</td>
</tr>
<tr>
<td>(k)</td>
<td>Thermal conductivity for a material (Btu/hr, ft, °F)</td>
</tr>
<tr>
<td>(I_{TL})</td>
<td>Time lag of a building material to heat flow (hours)</td>
</tr>
<tr>
<td>(q)</td>
<td>Rate of heat transfer (Btu/hr, (\text{ft}^2))</td>
</tr>
<tr>
<td>(Q)</td>
<td>Total cooling load (Btu)</td>
</tr>
<tr>
<td>(U)</td>
<td>Overall coefficient of heat transmission (Btu/hr, (\text{ft}^2), °F)</td>
</tr>
<tr>
<td>(F_D)</td>
<td>Ratios of incident solar energy to transmitted energy through glass for direct and diffuse solar radiations respectively, dimensionless</td>
</tr>
<tr>
<td>(S)</td>
<td>Shading coefficient (ratios of the solar heat gain for a particular type of fenestration to that for single unshaded double-strength sheet glass), dimensionless</td>
</tr>
</tbody>
</table>
MATHEMATICAL MODEL DEVELOPMENT

The basic methods followed for performing heat gain calculations are those found in Chapter 26 of the 1963 ASHRAE Guide and Data Book (Ref. 3). The calculational methods of the ASHRAE Tables, however, were found to be cumbersome for use on a computer, not general enough, and not sufficiently flexible because these tables are intended mainly for design conditions. Hence, more general and flexible mathematical equations for heat gain analysis by a computer had to be developed. The procedures of the ASHRAE Guide were employed, but the quantitative input was changed from that of looking up values in a table to one of a more general mathematical computation technique.

These techniques were developed in the form of equations such that computations could be made to determine:

- The instantaneous rate of heat gain
- The total cooling load at any instant
- The total yearly energy cost of the air-conditioning system

In order to determine the instantaneous heat gain and cooling load, several mathematical models had to be developed. The most complex models are listed below, and the development of each is under a separate heading in the following portions of this report. A summary in Appendix 1 shows the steps which must be followed in order to compute the actual cooling loads and costs. The specific models are:

- Sun position model
- Incident solar energy (direct)
- Incident solar energy (diffuse)
- Periodic heat flow (homogeneous and nonhomogeneous walls)
- Heat transfer through windows
- Energy costs of the air-conditioning system

Air-conditioning calculations involve many more considerations than are discussed in this report. For example, neither latent loads resulting from ventilated air and occupants nor other heat loads within the building are considered. Instead, emphasis has been placed on the problems where it is more difficult to obtain accurate values to represent the load, with particular attention placed on the complexities of the solar heat load. The sensible heat flow resulting from temperature difference of inside and outside air, solar energy, and periodic heat flow are also discussed. The derivations and the summary of the model equations which are discussed in the following sections are included in Appendix 1.

SUN POSITION MODEL

The method used to determine the sun's position in the sky relative to any specific point on the earth's surface was based on a graphical method. This method enables one to predict the sun's zenith angle (angle from sun to normal to earth's surface) and the sun's meridian angle (angle from North-South line to the sun's projection on the earth's surface).

In order to use this method on the computer, a simple process was used to transpose the graphical model of the sun's position into an exact mathematical model. To relate the equations more closely to the ASHRAE techniques and to simplify future computer programming, two minor differences were imposed. The equations were developed, 1) to compute the sun's altitude angle (angle from horizon) instead of its zenith angle, and 2) to compute the sun's azimuth angle (angle from true North) instead of its meridian angle which was measured from true South. This difference in the computation of the sun position is insignificant in that the altitude angle is merely the complement of the zenith angle, and the azimuth angle is the supplement of the meridian angle. The original sun position model with associated zenith and meridian angles is shown in Fig. 1.
The graphical device for computing the sun position angles is shown in Fig. 2. A description of the basis for construction of this diagram can be found in Ref. 4. The sun paths on the diagram were determined by a small semicircle with a radius which subtended a central angle in the larger circle of 23.5 degrees. This 23.5 degree angle is the angle between the celestial equatorial plane and the earth's equatorial plane. The angle also represents the limits of the sun's half cycle above the earth's equator and the half cycle below the earth's equator or the limits to the solar declination angles. The small semicircle representing the sun's half cycle is divided into six 30-degree sectors representing six months of the year, from June 21 to December 21. The other six months correspond exactly to the first six months as the sun travels through its reverse half cycle from December 21 to June 21.

To write equations to represent the geometry of the sun position diagram, the diagram was reconstructed as shown in Fig. 3. Angles were labeled on the reconstructed diagram so that the sun position could be defined relative to a point on the earth's surface and a specific time of day.

The equations for sun position are described below; their derivations are found in Appendix 1.

**Altitude angle** $\beta$, measured from horizon

$$\sin \beta = \sin \alpha \sin \phi - \cos \alpha \cos \phi \cos \left( \pi \frac{T}{12} \right)$$

where,

- $T$ = time of day $(00 < T < 24)$
  
  $T = 12$ = noon

- $\phi$ = latitude on earth (+ North, - South)

- $\alpha$ = solar declination angle
  $\alpha$ = $[-23.5^\circ, 23.5^\circ]$

- $d$ = number of days from June 21

  $d = 0$ = June 21; $d = 91$ = September 21

The angle $\pi T/12$ in the above expression for altitude angle is known as the solar hour angle. In this case it is measured from midnight. In many references this angle would contain the same terms but would be used to define the hour angle from solar noon, where $T = 0$ would indicate noon. In this report, however, the angle is used to define the hour angle from midnight as a matter of future convenience in writing computer programs. In this way the time of day can be steadily increased from 00 (midnight) to 24 (midnight next day) instead of being counted forward and backward from 12:00 noon where $T = 0$. The only difference this time definition makes in the equation is in the
sign inside the right hand side. For instance, in Refs. 7 and 21 this equation is written in the form:

\[
\cos \theta = \sin \alpha \sin \phi + \cos \alpha \cos \phi \cos h
\]

where, \(\cos \theta = \sin \beta\), because \(\beta = (\pi/2) - \theta\)

\(h = \) hour angle from solar noon

\(= (\pi T/12)\) for \(0 < T < 12\)

These two equations (1 and 1a) are identical because of the following relationship:

\[
\cos \pi T/12 = -\cos \pi T*/12
\]

\(*T = 0\) at noon \(*T = 12\) at noon

Azimuth angle (\(\psi_z\), measured from North)

\[
\cos \psi_z = \frac{\sin \phi \cos \alpha \cos (\pi T/12) + \cos \phi \sin \alpha}{\cos \beta}
\]

for \(|\cos \beta| > 0\)

Time of sunrise and sunset (\(T_R\), hours after midnight)

\[
T_R = \left(\frac{12}{\pi}\right) \cos^{-1} \left(\frac{\sin \alpha \sin \phi}{\cos \alpha \cos \phi}\right)
\]

Azimuth at sunrise (\(\psi_R\))

\[
\cos \psi_R = \frac{\sin \alpha}{\cos \phi}
\]

for \(|\sin \alpha| < |\cos \phi|\)

and \(-66.5^\circ < \phi < 66.5^\circ\)

Eqs. 1, 2, 3 and 4 are based on a circular orbit of the earth around the sun. Since the earth's orbit is not quite circular, but slightly elliptical, these values are not quite exact. The actual time values may vary from the calculated by as much as one quarter of an hour. If more accuracy is desired, this deviation can be corrected by applying a factor which expresses the time that the earth is leading or lagging the calculated times. The correction factors can be found in Ref. 4 and are also contained in The American Ephemeris and Nautical Almanac.

The times used in Eqs. 1, 2, 3 and 4 are the "mean solar times," referring to the computed times which give an average value to every hour and day. The eccentricity of the earth's orbit actually causes the earth to vary in speed as it travels about the sun. Since this variation in speed causes similar variations in the time that the sun will appear in a specific location, the term "apparent solar time" has been applied to the time system that is governed by the sun-earth relationships. This apparent solar time can be converted to mean solar time, and vice versa, by application of the equation of time, or "analemma." The apparent solar time is simply the mean solar time plus the equation of time.

The standard time on earth can also be calculated from the solar time using the equation:

Standard time = mean solar time + 4 (difference in longitude)

Standard time = apparent solar time - equation of time + 4 (difference in longitude)

The difference between the standard and solar times in these equations is expressed in minutes. The difference in longitude is expressed in degrees from the local "standard time meridian." Standard time meridians occur at intervals of 15 degrees around the earth (the angle through which the earth rotates each hour.)

<table>
<thead>
<tr>
<th>TABLE 1. Equation of Time in Minutes for the Days Shown (Ref. 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
</tr>
<tr>
<td>----</td>
</tr>
<tr>
<td>Jul.</td>
</tr>
<tr>
<td>Aug.</td>
</tr>
<tr>
<td>Oct.</td>
</tr>
<tr>
<td>Dec.</td>
</tr>
</tbody>
</table>

PROBLEM 1. Example Showing Exact Prediction of Sun Position

Find exact position of sun relative to a point at 42°N latitude and 85°W longitude on Feb. 1 at 9:00 a.m., standard time (ST).

Solution

Find apparent solar time (AST) using Eq. 5

\[
AST = ST + \text{equation of time (ET)} - 4 (\Delta \text{longitude})
\]

For Feb. 1, ET = +13.7 minutes from Table 1

Standard time meridian = 90°W longitude

\Delta \text{Longitude} = 85° - 90° = -5°

\(-5\) = -20 minutes

AST = 9:00 - 0:13.7 + 0:20 = 9:00 + 0:06.3

AST = 9:06.7 ≈ 9:06 a.m.
Find solar declination angle ($a$)

\[ a = \sin^{-1} \left[ \sin 23.5^\circ \cos \left( \frac{\pi d}{182.5} \right) \right] \]

$d = 143$ days from Feb. 1 to June 21

\[ a = \sin^{-1} \left[ 0.39875 \times (-0.776) \right] = \sin^{-1} [-0.31] \]

$a = -18^\circ$ (approximate)

or from Table 2

\[ a = -17^\circ 19' \]

Solar altitude angle ($\beta$)

\[ \sin \beta = \sin a \sin \phi - \cos a \cos \phi \cos \left( \frac{\pi T}{12} \right) \]

\[ a = -17.32^\circ, \ \phi = 42^\circ, \ T = 9.1 \text{ hours} \]

\[ \sin \beta = (-0.298) (0.67) - (0.955) (0.743) (-0.72) = -0.20 + 0.511 = 0.311 \]

Answer

\[ \beta = 18^\circ \text{ above horizon} \]

Azimuth angle ($\psi_z$)

\[ \cos \psi_z = \left[ \sin \phi \cos a \cos \left( \frac{\pi T}{12} \right) + \cos \phi \sin a \right] \frac{1}{\cos \beta} \]

\[ = \left[ (0.67) (0.955) (-0.72) + (0.743) (-0.298) \right] 1/0.95 = (-0.46 - 0.22)/0.95 = -0.68/0.95 = -0.716 \]

Answer

\[ \psi_z = 180 - 44 = 136^\circ \text{ from North} \]

Time of sunrise can also be calculated

\[ T_R = \frac{12}{\pi} \left[ \cos^{-1} (\tan a \tan \phi) \right] \]

\[ = \frac{12}{\pi} \left[ \cos^{-1} (-0.312) (0.9) \right] \]

\[ = 12/\pi \left[ \cos^{-1} (-0.281) \right] \]

\[ = 7.1 \text{ hours} = 7:06 \text{ a.m. AST} \]

Convert to standard time

\[ ST = \text{AST} + 0:06.3 = 7:06 + 0:06.3 \]

\[ = 7:12 \text{ a.m. ST} \]

---

**TABLE 2. Solar Declination ($a$) and the Ratio ($r$) of Solar Radiation Intensity Outside Earth's Atmosphere to Solar Constant (Ref. 5)**

<table>
<thead>
<tr>
<th>Mon.</th>
<th>1</th>
<th>8</th>
<th>15</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan.</td>
<td>-23°04'</td>
<td>1.0335</td>
<td>-22°21'</td>
<td>1.0325</td>
</tr>
<tr>
<td>Feb.</td>
<td>-17°19'</td>
<td>1.0288</td>
<td>-15°14'</td>
<td>1.0263</td>
</tr>
<tr>
<td>Mar.</td>
<td>-7°53'</td>
<td>1.0173</td>
<td>-9°11'</td>
<td>1.0140</td>
</tr>
<tr>
<td>Apr.</td>
<td>4°15'</td>
<td>1.0090</td>
<td>6°55'</td>
<td>0.9993</td>
</tr>
<tr>
<td>May</td>
<td>14°51'</td>
<td>0.9841</td>
<td>16°53'</td>
<td>0.9792</td>
</tr>
<tr>
<td>June</td>
<td>21°10'</td>
<td>0.9666</td>
<td>22°34'</td>
<td>0.9670</td>
</tr>
<tr>
<td>July</td>
<td>18°31'</td>
<td>0.9709</td>
<td>19°22'</td>
<td>0.9727</td>
</tr>
<tr>
<td>Aug.</td>
<td>5°34'</td>
<td>0.9882</td>
<td>9°59'</td>
<td>0.9862</td>
</tr>
<tr>
<td>Sept.</td>
<td>2°54'</td>
<td>0.9995</td>
<td>8°36'</td>
<td>1.0042</td>
</tr>
<tr>
<td>Oct.</td>
<td>-1°51'</td>
<td>1.0164</td>
<td>-10°21'</td>
<td>1.0207</td>
</tr>
<tr>
<td>Nov.</td>
<td>-2°41'</td>
<td>1.0288</td>
<td>-22°39'</td>
<td>1.0305</td>
</tr>
</tbody>
</table>

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**INCIDENT SOLAR ENERGY (DIRECT)**

The ASHRAE Guide (Ref. 3) gives various tables of values for estimating design loads on several different surface orientations. Since these values are for design purposes, they cannot be used for calculating actual conditions for year-round operating cost analysis. The use of tables is also restrictive because they lack the flexibility of use for many different surface orientations and for any position on earth. Thus a method was developed which would allow prediction of the incident direct solar energy at any time of day, any time of year, and any location on the earth. Some of the equations have been taken from outside sources; for the purposes of this report, however, these equations have been adapted to satisfy the special considerations necessary for computer programming.

**BASIC EQUATION**

The actual intensity of solar radiation that reaches the surface of the earth at a specific point on any given day is virtually impossible to predict. Collected solar data (Ref. 5), however, show that a calculation technique based on statistical analysis of solar radiation records can be quite accurate. Furthermore, it is not necessary to know the
weather condition at any one specific time to calculate the overall effect of the sun on the building heat gain. It is necessary, however, to be able to predict a "possible" weather condition so that certain solar irradiation values can be determined within the most probable bounds of statistical information. With the proper statistical data available, this estimation of solar conditions is quite feasible and accurate.

Before much solar radiation data were available, Moon (Ref. 6), in 1940, published his proposed standard solar-radiation curves for engineering use. Since that time there has been a great increase in the number of stations making continuous measurements of solar radiation, so that it has become possible to compare Moon's theoretical curves to measured values. In general, it has been found (Refs. 7, 8, 9, 10, and 11) that the measured values of solar radiation on clear days have exceeded Moon's standard by 10 and even up to 20 percent.

In any event, all of the equations that have been developed for the direct normal solar radiation have taken the same general empirical form from the plotting of various data.

\[
I_{DN} = I_0 e^{-a/cos \theta} z
\]

10 is a nearly constant value known as the "apparent solar constant." This term is created by a mathematical concept by extrapolating the atmospheric attenuation to zero. In reality, 10 has a value of about 85 percent of the true solar constant. The term "a" in the equation is denoted as the atmospheric extinction coefficient. The larger the value of "a," the less the solar radiation intensity will be that reaches the earth.

In developing Eq. 6, the logarithm of experimental values of \(I_{DN}\) is plotted vs. the secant of the zenith angle (i.e., 1/cos \(\theta_z\), also known as the air mass).

The author made several plots (see Fig. 8) using "Air Wall Data" (Ref. 20), and obtained the equation:

\[
I_{DN} = 370 e^{-0.161/cos \theta} z
\]

Moon's theoretical standard curve is approximately:

\[
I_{DN} = 360 e^{-0.220/cos \theta} z
\]

Majumdar and Kaushik (Ref. 10) used numerous experimental data and obtained an equation which has been rewritten in the form:

\[
I_{DN} = 391 e^{-0.204/cos \theta} z
\]

and is well within ±5 percent of all the measurements that they made.

The most thorough attempt to establish a definite equation is that of Stephenson (Ref. 7), who used Canadian data from the period 1960 through 1963. Only the best and consistent data were chosen from this period to form equations, and the days that were selected represented all seasons of the year. The equation that Stephenson established is of the form:

\[
I_{DN} = 375 e^{-a/cos \theta} z
\]

where "a" (the extinction coefficient) ranges from 0.08 in the winter to about 0.175 in the summer.

Even though from Stephenson's study some definite relationships can be seen between "a" and the time of year, it would be dangerous to try to use the equation for other locations on earth. The parameter "a" indicates the degree of haziness or cloudiness of the atmosphere and thus may be quite different for other locations, especially in southern latitudes where the winter and summer conditions may be completely reversed from the northern latitude conditions.

Liu and Jordan (Ref. 5) present an amazing statistical summary and analysis of solar radiation data from at least 27 different cities ranging from Alaska through many mainland states and as far south as Wake Island in the Pacific Area. The curves and equations developed in this reference are so convincing that full utilization of the work appears very reasonable.

The information presented in Ref. 5 is based on transmission coefficients and on a factor designated as the cloudiness index (Kp) to show the relative solar radiation contribution from the direct and sky (or diffuse) radiations. The transmission coefficient as defined in Ref. 5 is similar in concept to the extinction coefficient in Ref. 7. However, while Stephenson makes no attempt to distinguish the difference in the sky radiation between cloudy and clear days, Liu and Jordan provide curves showing that under some circumstances the overall transmission of direct radiation can be the same on a clear day as on a partly cloudy or hazy day; but the transmission of sky radiation will increase on cloudy days due to the additional scattering effects of the clouds.

The author deduced from Ref. 5 that the sky and atmospheric conditions can be fairly accurately predicted if one knows the total daily radiation on a horizontal surface, averaged for each month, and the total possible daily radiation on a horizontal surface. Most weather stations now make solar radiation measurements on a horizontal surface, and the total possible radiation can be calculated. Thus, all necessary information is available to predict, on a statistical basis, the condition of the sky and thus, the solar radiation contributions.
The condition of the atmosphere is specified by determining a value for "a" for use in Eq. 6. By using a statistical technique, the selected day can be chosen to be cloudy or clear, and a percent of cloudiness can be designated. The data available in Ref. 5 makes this statistical technique feasible.

The basic equation to be used for calculating instantaneous direct normal solar radiation is, therefore, of the form of Eq. 6. The value of "Iq" seems to be reasonably close to "375," and the value of "a" can be estimated when the atmospheric conditions are known.

PREDICTION OF ATMOSPHERIC CONDITION AND SOLAR RADIATION INTENSITY

As stated earlier, the actual atmospheric conditions are impossible to predict for any given day. A probable condition, however, can be predicted; hence, the overall solar effects over an extended period of time (e.g., one month) can be statistically predicted.

In order to describe the calculational technique, an example will be solved through the presentation and discussion of equations.

PROBLEM 2. Prediction of Direct Solar Radiation (Ref. 5)

Estimate the intensity of the direct solar radiation incident on a vertical wall facing southeast on January 16 at 11:30 a.m. (The problem of predicting the diffuse radiation will be discussed in the section on incident solar energy, diffuse.)

Information (generally available from most weather stations): The five year (1954-1958) average of the daily total radiation, H, on a horizontal surface for January in Indianapolis, Indiana (latitude 39°44') is 553 Btu/day, ft^2, according to data published by the U.S. Weather Bureau in Climatological Data, National Summary.

Solution

First, estimate the condition of the atmosphere by calculating a ratio, K_T, known as the monthly average cloudiness index.

K_T is equal to H, the monthly average total daily radiation on a horizontal surface, divided by H_o, the extraterrestrial daily radiation on a horizontal surface. The equation for K_T is:

\[ K_T = \frac{H}{H_o} \]  

H_o is calculated as the mean value in the month using January 16, the middle of the month. The equation for H_o is:

\[ H_o = \frac{24}{\pi} r_l \frac{1}{\cos \phi \cos \alpha} \sin \left( \frac{\pi}{12} - \frac{\pi}{12} \right) \]

where, r_l is the extraterrestrial solar radiation intensity normal to the sun's rays; actually the value represents the effective solar constant for a specified time of year.

r is an adjustment to the solar constant (I_s) and can be obtained from Table 2.

For over 50 years, the Smithsonian Institute has measured the value of the solar constant and has established its value with a probable error of 2 percent. Table 3, taken from Ref. 21, gives the value in several units.

<table>
<thead>
<tr>
<th>Unit</th>
<th>langley/min</th>
<th>Btu/hr, ft^2</th>
<th>w/m^2</th>
<th>w/ft^2</th>
<th>kwh/day, ft^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>2.00</td>
<td>442.4</td>
<td>1395.0</td>
<td>129.6</td>
<td>3.11</td>
</tr>
</tbody>
</table>

Problem 2 requires units to be in Btu/hr, ft^2; from Table 3, this unit value is seen to be 442.4. Table 2 gives the value of r for January as 1.0315; therefore,

\[ r_l = 456 \text{ Btu/hr}, \text{ft}^2 \]

The solar declination angle (\( \alpha \)) on January 16 is -21°16', from Table 2.

The time of sunrise (T_R) for this part of the problem can be computed as the mean solar time from Eq. 3, using \( \alpha = -21.67° \) and \( \phi = 39.73° \)

\[ T_R = 7.27 \text{ hours or 7:16 a.m.} \]

From Eq. 8, H_o is 1350 Btu/day, ft^2; therefore, \( K_T = \frac{H}{H_o} = \frac{553/1350}{0.41} \). In some instances a weather station may already have the K_T value available, indicating the ratio of actual radiation received to the maximum amount possible.

The distribution of the cloudiness index (K_T) values for all days during the month can now be estimated by using K_T and the probability distribution curves in Ref. 5. Liu and Jordan have been able to correlate data from widely dispersed locations into well defined curves of different K_T values. They have shown, for example, that for a (K_T = 0, 6), the fractional distribution of different conditions is the
same for the month regardless of the location on earth. With this information they have provided a graph containing "generalized monthly Kp curves." Using the Kp values, the overall solar radiation effects on the earth can be estimated. The details of the atmosphere—whether cloudy, hazy, or industrial—is not important, since the overall effect would be the same from any of these situations.

It is now necessary to estimate the "type of day" for January 16. (Actually the specific date does not matter. January 5 has the same chance of being cloudy as does January 16.) This estimation is done by selecting a random number (between 0 and 1) and using it in the probability distribution curves. The more numbers that are chosen the better the statistics will become on the estimations.

(This technique of estimation is known as the Monte Carlo technique. It is actually a nontheoretical approach, because, in effect, it is an experiment using the physical laws of nature. Therefore, in many cases it can be more accurate than a more theoretical approach if the numbers of statistics are large. The computer is very adept at solving this type of problem and is also unbiased in the selection of random numbers.)

Suppose one selects the number "0.8." Then using Table 4 and the column of numbers under Kp = 0.4, one establishes the location of the value, "0.8." This occurs adjacent to Kp = 0.63. Therefore, for one particular day during the month the cloudiness index (Kt) is equal to 0.63. It can be shown that as more and more random numbers are selected and the Kt numbers are established, one would see a tendency toward a certain group of Kt numbers. This prediction technique will not produce an accurate estimate of the Kt value for one given day; however, over a longer period of time (a month) the selection of Kt values will average around Kt because the selection process is actually governed by the Kt curve.

An estimation of the percent of cloud cover can be obtained from the correlation curve developed by Becker and Boyd (Ref. 12). It has been rewritten in the form:

\[ K_c = 1.6 (1 - K_T) \]  

where \( K_c \) = fraction of cloud cover. When \( K_T \) is less than 0.38, the cloud cover is 100 percent.

Using the value of \( K_T = 0.63 \), the value of \( K_c \) becomes 0.59.

Thus, the sun is shaded by clouds 59 percent of the entire day and is shining 41 percent of the day. The selection of random numbers for calculation of solar conditions throughout the day will predict a near 59 percent cloudy condition and a 41 percent sunshine condition. Again, the more random numbers that are selected, the closer the conditions will approach the 59 to 41 percent ratio.

At this point another random number is chosen, say 0.45. Since it is less than the \( K_c \) value of 0.59, then one would say that the sun is covered by clouds. (If, instead of 0.45, 0.71 were chosen, then one would say that the sun is shining, because 0.71 is larger than 0.59 and thus falls within the 49 percent of the probable random selections.)

Depending upon which sky condition exists, cloudy or clear, one of the following equations (Ref. 5) should be used to determine the percentage of total radiation that is due to direct only. The other fraction

<table>
<thead>
<tr>
<th>K_T</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.073</td>
<td>0.015</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.08</td>
<td>0.162</td>
<td>0.070</td>
<td>0.023</td>
<td>0.008</td>
<td>0.000</td>
</tr>
<tr>
<td>0.12</td>
<td>0.245</td>
<td>0.129</td>
<td>0.045</td>
<td>0.021</td>
<td>0.007</td>
</tr>
<tr>
<td>0.16</td>
<td>0.356</td>
<td>0.190</td>
<td>0.082</td>
<td>0.039</td>
<td>0.007</td>
</tr>
<tr>
<td>0.20</td>
<td>0.456</td>
<td>0.249</td>
<td>0.121</td>
<td>0.053</td>
<td>0.007</td>
</tr>
<tr>
<td>0.24</td>
<td>0.546</td>
<td>0.298</td>
<td>0.160</td>
<td>0.076</td>
<td>0.013</td>
</tr>
<tr>
<td>0.28</td>
<td>0.653</td>
<td>0.346</td>
<td>0.194</td>
<td>0.101</td>
<td>0.013</td>
</tr>
<tr>
<td>0.32</td>
<td>0.759</td>
<td>0.379</td>
<td>0.224</td>
<td>0.126</td>
<td>0.013</td>
</tr>
<tr>
<td>0.36</td>
<td>0.862</td>
<td>0.438</td>
<td>0.277</td>
<td>0.152</td>
<td>0.027</td>
</tr>
<tr>
<td>0.40</td>
<td>0.967</td>
<td>0.493</td>
<td>0.323</td>
<td>0.191</td>
<td>0.034</td>
</tr>
<tr>
<td>0.44</td>
<td>1.074</td>
<td>0.545</td>
<td>0.358</td>
<td>0.235</td>
<td>0.047</td>
</tr>
<tr>
<td>0.48</td>
<td>1.179</td>
<td>0.601</td>
<td>0.400</td>
<td>0.269</td>
<td>0.054</td>
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<td>0.52</td>
<td>1.278</td>
<td>0.654</td>
<td>0.460</td>
<td>0.310</td>
<td>0.081</td>
</tr>
<tr>
<td>0.56</td>
<td>1.376</td>
<td>0.719</td>
<td>0.509</td>
<td>0.350</td>
<td>0.128</td>
</tr>
<tr>
<td>0.60</td>
<td>1.474</td>
<td>0.760</td>
<td>0.561</td>
<td>0.410</td>
<td>0.161</td>
</tr>
<tr>
<td>0.64</td>
<td>1.572</td>
<td>0.827</td>
<td>0.673</td>
<td>0.467</td>
<td>0.228</td>
</tr>
<tr>
<td>0.68</td>
<td>1.669</td>
<td>0.888</td>
<td>0.792</td>
<td>0.538</td>
<td>0.295</td>
</tr>
<tr>
<td>0.72</td>
<td>1.767</td>
<td>0.931</td>
<td>0.873</td>
<td>0.648</td>
<td>0.377</td>
</tr>
<tr>
<td>0.76</td>
<td>1.864</td>
<td>0.967</td>
<td>0.945</td>
<td>0.758</td>
<td>0.478</td>
</tr>
<tr>
<td>0.80</td>
<td>1.961</td>
<td>0.981</td>
<td>0.980</td>
<td>0.884</td>
<td>0.595</td>
</tr>
<tr>
<td>0.84</td>
<td>2.057</td>
<td>0.997</td>
<td>0.993</td>
<td>0.945</td>
<td>0.714</td>
</tr>
<tr>
<td>0.88</td>
<td>2.152</td>
<td>0.999</td>
<td>1.000</td>
<td>0.985</td>
<td>0.840</td>
</tr>
<tr>
<td>0.92</td>
<td>2.247</td>
<td>0.999</td>
<td>1.000</td>
<td>0.999</td>
<td>1.000</td>
</tr>
<tr>
<td>0.96</td>
<td>2.341</td>
<td>1.000</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1.00</td>
<td>2.341</td>
<td>1.000</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
of radiation is due, of course, to the diffuse contribution discussed in the section on incident solar energy, diffuse.

Clear

\[ K_D = 1.415 K_T - 0.384 \]  
for \( 0.38 \leq K_T \leq 0.80 \)

\[ K_D = 0.75 \]  
for \( K_T > 0.80 \)

Cloudy

\[ K_D = 1.492 K_T - 0.492 \]  
for \( 0.60 \leq K_T \leq 0.83 \)

\[ K_D = e^{(0.935 K_T^2)} - 1.0 \]  
for \( K_T < 0.60 \)

For Problem 2 where \( K_T = 0.63 \), the clear part of the day will provide direct solar radiation according to

\[ K_D = 1.415 (0.63) - 0.384 = 0.508 \]

and during the cloudy part of the day according to

\[ K_D = 1.492 (0.63) - 0.492 = 0.448 \]

The atmospheric extinction coefficient (\( a \)) can be estimated by equating the direct solar radiation equation to the average daily intensity of direct solar radiation. The average value is given by

\[ I_{DH}^{(ave)} = \frac{K_D H}{2(12 - T^)} \]

For Problem 2,

\[ I_{DH}^{(ave)} = 72.5 \text{ Btu/hr, ft}^2 \], if clear

\[ I_{DH}^{(ave)} = 64 \text{ Btu/hr, ft}^2 \], if cloudy

It was found by plotting several daily solar radiation curves that this average value of \( I_{DH} \) on a horizontal surface is about 0.6 of the maximum value of the direct radiation on a horizontal surface. The equation for radiation on a horizontal surface is:

\[ I_{DH} = r I_o e^{-a/cos \theta} z \cos \theta \]

This can also be expressed as:

\[ I_{DH} = r I_o e^{-a/sin \beta} \sin \beta \]

therefore, the maximum value will occur where \( \beta \) (the solar altitude) is maximum.

\[ \beta_{max} = \left( \frac{\pi}{2} \right) - \phi + a \]

In this example \( \beta_{max} = 90^\circ - 39.73^\circ + (-21.67^\circ) = 28.6^\circ \)

therefore, \( \sin \beta_{max} = 0.48 \).

The value for "\( a \)" can now be calculated by the formula:

\[ a = -\sin \beta_{max} \log \left( \frac{I_{DH}^{(ave)}}{225r \sin \beta_{max}} \right) \]

For Problem 2,

\[ a = 0.207 \text{ for a clear sky} \]

\[ a = 0.267 \text{ for a cloudy sky} \]

**Final Equations** (for January 16)

\[ I_{DN} = r I_o e^{-0.207/sin \beta} \text{, for a clear sky} \]

\[ I_{DN} = r I_o e^{-0.267/sin \beta} \text{, for a cloudy sky} \]

\[ r I_o = 1.0315 (375) = 387 \text{ Btu/hr, ft}^2 \]

\[ I_{DH} = \frac{I_{DN} \sin \beta}{r} \]

For vertical surface of a wall, the direct solar radiation is the product of the incident normal radiation times the cosine of the incident angle to the wall's normal

\[ I_{DV} = I_{DN} \cos \theta \]

where \( \cos \theta = \cos \beta \cos \psi_z - \sin \psi \psi_N \)

\[ I_{DV} = I_{DN} \cos \beta \cos \psi_z - \psi_N \]

\[ I_{DV} = I_{DN} \cos \beta \cos \psi_z - \psi_N \]
INCIDENT SOLAR ENERGY (DIFFUSE)

Stephenson's records for clear days (Ref. 7) show that the diffuse radiation from sky scattering is directly proportional to $I_{DN}$ and that the proportionality constant $B$ can be related to the atmospheric extinction coefficient. The graph of $B$ vs. $a$ in his report, however, shows quite a scattering of points about the straight line of $B = 0.67a$.

With the available information in Ref. 5 which presents graphs of the relationships between the diffuse and direct solar radiation, this value of $B$ can be improved. $B$ can then be used to compute the instantaneous diffuse solar radiation on a horizontal surface according to the equation:

$$I_{dh} = B I_{DN}$$  \hspace{1cm} (20)

The value of $B$ can be shown to vary slightly and can be defined as:

$$B = \left( \frac{r_d}{r_D} \right) \sin \beta$$  \hspace{1cm} (21)

where $r_d$ and $r_D$ are the transmission coefficients for the diffuse and direct solar radiations, respectively. The transmission coefficients represent the ratio of the incident solar radiation on a horizontal surface to the extraterrestrial solar radiation on a horizontal surface. Therefore, $r_D$ can be computed simply as the fraction:

$$r_D = \frac{I_{DN}}{I_{SC}}$$  \hspace{1cm} (22)

The value for $r_d$ depends upon the cloudiness of the sky. Ref. 5 presents graphs of $r_d$ vs. $r_D$ which yield the following equations for $r_d$:

**Clear**

$$r_d = 0.2710 - 0.2939 r_D$$  \hspace{1cm} (23)

**Cloudy**

$$r_d = 0.33 (1 - r_D)$$  \hspace{1cm} (24)

for $0.4 \leq r_D$

$$r_d = \sqrt{1.07 \log \left( \frac{r_D + 1}{r_D} \right)} - r_D$$  \hspace{1cm} (25)

for $r_D < 0.4$

Threlkeld (Ref. 11) has shown that the diffuse radiation on a vertical surface is related both to the magnitude of the diffuse radiation on a horizontal surface and to the angle of incidence of the sun ray on the vertical surface. In effect, this says that the diffuse radiation on a surface depends on what part of the sky the surface "sees" relative to the position of the sun.

According to Ref. 7, the diffuse radiation intensity on a vertical surface can be computed by the formula:

$$I_d = I_{dh} F(\cos \theta)$$  \hspace{1cm} (26)

The function of $\cos \theta$ from Threlkeld's paper is given by the following expression:

$$F(\cos \theta) = 0.55 + 0.437 \cos \theta + 0.313 \cos^2 \theta$$  \hspace{1cm} (27)

for $\cos \theta > -0.2$

and

$$F(\cos \theta) = 0.45$$  \hspace{1cm} (28)

for $\cos \theta < -0.2$

The distribution should change slightly from the condition when the sky is clear to when it is cloudy. The above functions do not predict any change. The distribution of the diffuse solar radiation from cloudy skies on vertical surfaces would differ only in magnitude (larger) from the clear sky values according to the above equations. Because of the complexities in establishing a more exact equation for the diffuse values, further refinement was not worthwhile. Furthermore, the percentage of the total solar load due to diffuse radiation is on the order of only 10 to 15 percent; therefore, slight errors in the approximation may be permissible since small errors will not be significant in the total load calculation.

**PROBLEM 3. Prediction of Diffuse Solar Radiation**

Solve for the diffuse and total solar radiation intensity on the southeast wall of the building in Problem 2, for clear and cloudy sky conditions.

**Solution**

Solve for transmission coefficient ($r_d$)

$$r_D = \frac{I_{DN}}{I_{SC}} = \frac{250}{387} = 0.646$$

$$\therefore r_d = 0.2710 - 0.2939 (0.646) = 0.08$$, if clear

$$r_D = \frac{219}{387} = 0.566$$

$$\therefore r_d = 0.33 (1 - 0.566) = 0.143$$, if cloudy
Solve for $B$

$$B = \frac{r}{r} \sin \beta = 0.08/0.646 (0.47)$$

$= 0.058$, if clear

$B = 0.143/0.566 (0.47)$

$= 0.119$, if cloudy

Solve for $I_{dh}$

$$I_{dh} = B_{DN} = 0.058 (250)$$

$= 14.5$ Btu/hr, ft$^2$, if clear

$I_{dh} = (0.119) (219)$

$= 26$ Btu/hr, ft$^2$, if cloudy

Solve for $F (\cos \theta)$

$$\cos \theta = \cos \beta \cos \left| \Psi_s - \Psi_N \right| = 0.705$$

$F (\cos \theta) = 0.55 + 0.437 \cos \theta + 0.313 \cos^2 \theta$

$= 0.55 + (0.437)(0.705) + (0.313)(0.5)$

$= 0.55 + 0.308 + 0.156 = 1.014$

Answer to Problem 3, diffuse

$I_d = 1.014 (14.5)$

$= 14.7$ Btu/hr, ft$^2$, if clear

$I_d = 1.014 (26)$

$= 26.4$ Btu/hr, ft$^2$, if cloudy

Answer to Problem 3, total radiation

$I = 176 + 14.7$

$= 190.7$ Btu/hr, ft$^2$, if clear

$I = 155 + 26.4$

$= 181.4$ Btu/hr, ft$^2$, if cloudy

PERIODIC HEAT FLOW THROUGH HOMOGENEOUS AND NONHOMOGENEOUS WALLS

In order to compute the amount or rate of transfer of heat into the material on the surface of a building, it is necessary to know:

- The intensity of direct solar radiation striking the surface
- The absorptivity (or reflectivity) of the surface for direct solar radiation
- The intensity of diffuse solar radiation striking the surface
- The absorptivity (or reflectivity) of the surface for diffuse solar radiation
- The rate at which the surface emits radiation to the sky and other surroundings
- The rate at which the surface absorbs the low temperature radiation emitted by the sky and other surroundings by virtue of their temperatures and radiating characteristics
- The temperature of the surrounding air
- The temperature of the outer building surface
- The unit convective conductance for heat transfer between the air and the building surface

The method of heat gain analysis used in the ASHRAE Guide simplifies the complex interrelationships of the above factors through the use of the sol-air temperature concept. The sol-air temperature is that temperature of the outdoor air, which, in the absence of all radiation exchanges, would give the same rate of heat flow into the surface as would exist with the actual combination of incident solar radiation, radiant energy exchange with the sky and other outdoor surroundings, and convective heat exchange with the outdoor air.

Several tables were developed in the ASHRAE Guide which provide a total equivalent temperature differential which, when applied to an overall U-factor of a wall, can be used to compute the total heat transmission. The use of the total equivalent temperature differential has several distinct advantages for hand calculations and for computing design loads; however, this method is not flexible enough for year-round heat gain analysis by the computer and does not supply time lag information.

The basis for the development of the total equivalent temperature differential values in the ASHRAE Guide is the sol-air temperature data developed by Mackey and Wright around 1944 or 1945. Before that time (1943) Mackey and Wright had developed an analytical approach for the computation of heat flow through walls and roofs and associated
time lags. Their work is reported in Refs. 13 and 14. These references were studied, and the method of heat flow analysis appeared to be very applicable to digital computation. The method of Mackey and Wright was, therefore, used in the development of a mathematical model of heat flow.

Actually, the heat flow could be calculated more accurately if the true outside surface temperature of the wall or roof could be established, instead of using the fictitious values of the sol-air temperature. This requires knowledge of the surface materials' reflectance, absorptance, and emittance and a computation technique which would be based on heat flow equilibrium at the outside surface.

In future developments of this model and computer programs which are to follow, the above method will probably be pursued to greater lengths and employed in a more accurate calculational model. Once the outside surface temperature is determined, the Mackey and Wright model may be used to determine the amount of heat flow that occurs through the wall and roof materials and the associated time lags.

Appendix 1 describes the development of the heat flow equations by Mackey and Wright, and the most important equations are listed below. The terms in these equations are described in more detail in Appendix 1.

**Sol-Air Temperature** ($t_e$)

\[
t_e = t_o + \frac{bI}{h_L}
\]

where, $t_o =$ outdoor dry-bulb temperature of the air  
$b =$ solar absorptivity of surface  
$I =$ total incident solar energy in Btu/hr, ft$^2$  
$h_L =$ outdoor air film coefficient in Btu/hr, ft$^2$, °F

**Temperature of Wall's Inside Surface** ($t_o$)

\[
t_o = t_o^M + \lambda_e (t_e - t_m)
\]

where, $t_o =$ sol-air temperature at time T  
$t_o^M =$ inside wall surface temperature at time $T + TL$  
$T =$ specified time of day in hours  
$TL =$ time lag of wall, indicating that the temperature of the inside surface of a wall at time $T + TL$ is related to the sol-air temperature outside at time $T$  
$t_e =$ steady-flow mean daily temperature of the inside surface of the wall  
$t_m =$ the daily average sol-air temperature  
$\lambda_e =$ mean decrement factor using the first and second harmonic

The value for the steady-flow mean daily temperature ($t_M$) can be computed from the equation:

\[
t_M = t_i + \frac{1}{h_o} \left( t_m - t_i \right) + \frac{1}{h_o + 1/h_L + L/k}
\]

where, $h_o =$ inside air film coefficient in Btu/hr, ft$^2$, °F  
$t_i =$ temperature of inside air  
$I/o =$ resistance of inside air film  
$L/k =$ resistance of wall or roof materials

The value for $h_o$ (the inside air film coefficient) is usually 1.65 Btu/hr, ft$^2$, °F. The outside air film coefficient is usually around 4, but varies somewhat with the external wind velocity. From the data of Mackey and Wright for a brick wall surface, the value of $h_L$ would vary as:

\[
h_L = 2.0 + 0.4 (V_w) \text{ Btu/hr, ft}^2, \text{ °F}
\]

where $V_w =$ the velocity of the wind parallel to the surface of the wall.

**Heat Transfer** ($q$)

\[
q = h_o (t_o - t_i) \text{ Btu/hr, ft}^2
\]
HEAT TRANSFER THROUGH WINDOWS

The method used to calculate the heat transfer through windows is the same as that used in the 1963 ASHRAE Guide and Data Book, with the recent revisions published in a recent ASHRAE paper (Ref. 15). The basic heat gain equation for windows is:

\[
q = S \left( \frac{\text{Solar Heat Gain Factor}}{U (t_a - t_i)} \right) \text{Btu/hr ft}^2 \tag{34}
\]

\( S \) in Eq. 34 is the shading coefficient, a multiplying factor relating the solar heat gains for any type of glass and glass-shading combination to the solar heat gains for 1/8 inch sheet glass. This constant multiplier for each type glass can be used because the recent ASHRAE studies (Ref. 15) have shown that "the ratio of solar heat gains for all types of single and insulating glass and for many types of glass and shading combinations remains reasonably constant for all incident angles and intensities of the sun."

Values of solar heat gain factors are given in the 1963 ASHRAE Guide, but are improved for more general use in Ref. 15. In this reference ASHRAE has developed curves for the transmittance and absorbance of solar energy vs. the incident angle of the sun's rays. These curves were drawn for 1/8 inch sheet glass without shading. This was done for convenience since this glass has the highest solar heat gain of any commonly used type, and therefore shading coefficients for all other glass types will have values less than one. The equation for the solar heat gain factor is:

\[
\text{Solar Heat Gain Factor} = \frac{\text{Transmitted Energy}}{\text{Absorbed Energy}} + 0.27 \tag{35}
\]

where, 0.27 indicates that 27 percent of the absorbed energy is transferred to the indoors, and

\[
\text{Transmitted Energy} = (F_D I_{DN} + F_d I_d) \text{Btu/hr ft}^2 \tag{36}
\]

The \( F_D \) and \( F_d \) terms represent the ratios of solar heat gain to the incident energy for direct and diffuse respectively, where, according to the curves in Ref. 15,

\[
F_d = 0.82 \tag{37}
\]

\[
F_D = 0.865 e^{-0.019/cos^2 \theta} \tag{38}
\]

The maximum value for Eq. 38, for normal incidence, is 0.85. This equation was developed by plotting the data from Ref. 15 on a graph showing log \( F_D \) vs. \( 1/cos^2 \theta \).

The values for the absorbed energy are 0.02 for diffuse and approximately 0.0236 \( F_D \) for direct radiation. Therefore, the equation for absorbed energy is:

\[
\text{Absorbed Energy} = 0.0236 F_D I_{DN} + 0.02 I_d \tag{39}
\]

DETERMINATION OF OPERATING COSTS

In order to compare the costs between building systems—in this case each system consists of air-conditioning equipment and wall and roof assemblages—it is necessary to establish an ultimate cost of each system combination. The establishment of all of the costs which make up the total ultimate cost of such system combinations can be quite involved, if many different types of air-conditioning systems are to be considered. Other studies (Refs. 16 and 17) have evolved methods for cost comparisons of various heating and cooling systems. However, this report is concerned only with establishing a more reliable operating cost. This operating cost would consist of all costs, associated with both the air-conditioning system and the wall and roof systems, which indicate the differences between the systems. Those differences (consisting of yearly fuel costs, maintenance costs, repair, window upkeep and replacement) would be the basis for making a choice of one system combination over another. In light of this objective the following assumptions are made as a basis for determining the differences in costs between various systems:

\[\begin{itemize}
  \item That the variables in costs between different wall and roof systems are in (a) initial cost, (b) installation cost, (c) maintenance costs, including washing and repair, (d) heating and cooling costs which result from heat gain and heat loss through the wall and roof assemblages, and (e) the type of air-conditioning system used.
  \item That the selection of a wall and/or roof system can be optimized by minimizing both the total costs listed in the above item and the cost differences incurred because of interest on money.
  \item That items a, b, and c in the first item are available from various construction and manufacturer records; that thermal properties of various wall and roof assemblages can be obtained; and that d can be calculated through the use of the models developed in this report.
\end{itemize}\]
In addition to the above assumptions, one more equation is necessary in order to be able to compare one system with another on a cost basis; i.e., the calculation of the cost of the operation of the air-conditioning system. In this instance it is the cooling cost that would result from a difference between the building envelope systems.

This difference would occur not only as a result of the solar loads but also from the indoor-outdoor temperature differentials. The proportion of these loads, of course, will depend on the exposures of the building walls, i.e., the orientation and the division of the cooling zones within the building. Using the sum of these two external sources of heat, the total yearly load can be established by totalling the daily loads, with one exception—on certain days when the outdoor temperatures remain below 55°F, the outdoor air can be used to cool the interior of the building, instead of using electrical power for this purpose. In this case the heat gains will not be added to the total cooling load.

In addition to summing the cooling loads over a year's period, a cost per unit of heat removed must be established. This cost will depend upon the type of air-conditioning system used, its efficiency, and the local cost of fuel (or power) that is used. The computation for total energy cost \( C_T \) would be:

\[
C_T = RQ
\]

(40)

where,

- \( R \) = unit cost per unit of heat removed
- \( Q \) = total cooling load (Btu)

Figure 5 shows a plot of \( C_T \) (total cost) vs. \( Q \) (the total load) for different values of \( R \).

**FIG. 5.**
Cooling Energy Costs vs. Total Heat Gains

| \( C_T \) = total energy cost in thousands of dollars |
| \( Q \) = total cooling load in billions of BTU's |

To make the optimum selection of building wall and roof assemblages, the yearly operating costs of the air-conditioning system should be determined. To accomplish this, more than just the design conditions for the heat gains and heat losses are necessary. The actual weather and solar conditions must be simulated and integrated over a long period of time. This simulation is necessary because the many variables which occur under the solar load calculations cannot be generalized. One of these variables which is unique to each building is the shading effect derived from adjacent buildings or natural effects of terrain. Another variable is the building shape and its orientation. These effects may sometimes be simplified to a certain degree; but, until such effects reveal consistent and predictable trends, a precise method of calculation should be employed.

The intent in this study was to establish as precise a method of calculation as possible with the use of the best available methods and data which now include conclusive data on the establishment of solar radiation on horizontal surfaces. Using these data along with other procedures which describe sky conditions (given horizontal radiation information), the author feels that the equations developed in this report will yield fairly accurate results in solar load and heat flow calculations.

On the other hand, just as important as the development of the equations is the manner in which they are used. The Monte Carlo method of simulating the solar conditions appears to be just as applicable to the methods that are presented in this report as it is to the problems in nuclear science. The Monte Carlo method has been used with remarkable success in the nuclear science field and has recently been employed to simulate auditorium acoustics (Ref. 18). It is based on physical processes and, on the basis of a statistical random selection of numbers, simulates a situation in very much the same way as nature does.

Future computer programming to test the equations in this report will employ the Monte Carlo technique for solar radiation prediction. The accuracy of these predictions will depend on the amount of statistics generated. It may be necessary to actually simulate five years of operation or only one year of operation, depending on how the results compare to actual solar radiation data.

As for heat flow calculations, the methods of Mackey and Wright have already been tested in several computer programs written for the IBM 7074, and results agree favorably with actual test data. These methods are also receiving extensive use in computer programs for air-conditioning load calculations presently being used by a major organization (Ref. 19) for mechanical engineering consulting work.
Many calculations using the solar radiation equations for clear days have been compared to incident solar radiation data obtained at the Air Wall Research Building (Ref. 20) on The Pennsylvania State University campus. These results showed fairly close comparisons—even the diffuse values. However, comparison of any of the equations to a specific day in the year is not evidence of a valid computational model. More important is: will the total calculated loads compare with actual loads over a period of a month or so with a high degree of accuracy? Only when this comparison is made will the model be regarded as a valid computational tool.

CITED REFERENCES


11. Threlkeld, J. L., Solar Irradiation of Surfaces on Clear Days; ASHRAE J., vol. 4, no. 11, Nov. 1962, pp. 43-54


16. The Economics of Sensible Heat Control, AIA File, no. 30-A, Owens-Corning Fiberglas Corporation, Feb. 1963


UNCITED REFERENCES


APPENDIX 1. DERIVATION OF EQUATIONS

SUN POSITION

The following equations are derived by using the reconstructed diagram (Fig. 3) where the physical dimensions are represented by letters.

First, find zenith angle ($\theta_z$)

\[ R = 1 \]
\[ r = R \sin 23.5^\circ = (1) (0.39875) = 0.39875 \]
\[ r' = 0.39875 \cos \left( \frac{\pi d}{182.5} \right) = \sin \alpha \]
\[ d = \text{number of days from June 21, where } 0 < d < 182.5 \]
\[ \alpha = \text{latitude angle} \]
\[ T = \text{absolute value of time of day in hours away from noon} \]
\[ \text{e.g., } T = 0 \text{ at noon, and } T = 1 \text{ for both 11 a.m. and 1 p.m.} \]
\[ \beta' = \cos \alpha \left[ 1 - \cos \left( \frac{\pi T}{12} \right) \right] \]

therefore, the first relation is established:

\[ y = 1 - \sin \phi \sin \alpha - \cos \phi \cos \alpha \cos \left( \frac{\pi T}{12} \right) \]

The second relation by observation is:

\[ y = 1 - \cos \theta_z \]

The two relations can be equated as follows:

\[ 1 - \cos \theta_z = 1 - \sin \phi \sin \alpha - \cos \phi \cos \alpha \cos \left( \frac{\pi T}{12} \right) \]

therefore, the following three equations result:

\[ \text{Zenith angle (for } T = 0 \text{ at noon)} \]
\[ \cos \theta_z = \sin \phi \sin \alpha + \cos \phi \cos \alpha \cos \left( \frac{\pi T}{12} \right) \]

\[ \text{Zenith angle (for } T = 12 \text{ at noon)} \]
\[ \cos \theta_z = \sin \phi \sin \alpha - \cos \phi \cos \alpha \cos \left( \frac{\pi T}{12} \right) \]

Altitude angle ($\beta$)
\[ \sin \beta = \cos \theta_z \]

Second, find azimuth angle ($\psi_z$)

Let $R' = \sin \theta_z$
\[ h = \sin \beta = \sin (\phi - \alpha) \]

therefore, the first relation is established:

\[ h = -b' \sin \phi \]

The second relation is:

\[ \cos \psi_z = \sin \theta_z \cos \psi_z \]

The two relations can be equated as follows:

\[ h = b' \sin \phi = \sin \theta_z \cos \psi_z \]

\[ \cos \psi_z = \frac{(h - b' \sin \phi)}{\sin \theta_z} \]

but,
\[ b' = \cos \alpha \left[ 1 - \cos \left( \frac{\pi T}{12} \right) \right] \]

therefore,
\[ \cos \psi_z = \frac{\sin (\phi - \alpha) - \cos \alpha \sin \phi + \cos \alpha \sin \phi \cos \left( \frac{\pi T}{12} \right)}{\cos \beta} \]

by trigonometric identities, it can be shown that:

\[ \text{Azimuth angle (for } T = 0 \text{ at noon and } \psi_z = 0 \text{ at south)} \]
\[ \cos \psi_z = \frac{\sin \phi \cos \alpha \cos \left( \frac{\pi T}{12} \right)}{\cos \beta} \]

\[ \text{Azimuth angle (for } T = 12 \text{ at noon and } \psi_z = 0 \text{ at south)} \]
\[ \cos \psi_z = \frac{\sin \phi \cos \alpha \cos \left( \frac{\pi T}{12} \right) + \cos \phi \sin \alpha}{\cos \beta} \]

Aximuth angle (for $|\beta| = \pi/2$)
\[ \psi_z = 0 \]

Third, find sunrise and sunset times ($T_R$)

Known:
\[ \beta = 0 \]
\[ \sin \beta = 0 = \sin \phi \sin \alpha - \cos \phi \cos \alpha \cos \left( \frac{\pi T_R}{12} \right) \]
\[ \cos \left( \frac{\pi T_R}{12} \right) = \frac{\sin \phi \sin \alpha}{\cos \phi \cos \alpha} \]

Time of sunrise (for $T_R$ measured in hours from midnight)
\[ T_R = \left(12/\pi\right) \cos^{-1} \left[ \frac{\sin \phi \sin \alpha}{\cos \phi \cos \alpha} \right] \]
Fourth, find azimuth angle at sunrise ($\psi^w_R$).

Set $\beta = 0$, $cos \beta = 1$

From the equations for $\psi^w_z$ and $T^w_R$

$$\cos \psi^w_R = \sin \phi \cos a + \cos \phi \sin a$$

$$\cos \psi^w_R = \frac{\sin \phi \sin a + \cos \phi \sin a}{\cos \phi}$$

Azimuth angle at sunrise (measured from north)

$$\cos \psi^w_R = \frac{\sin a}{\cos \phi}$$

INCIDENT SOLAR ENERGY (DIRECT)

The form of the equation for incident direct normal solar energy is:

$$I_{DN} = I_0 e^{-a/\cos \theta_z}$$

which is obtained from plotting a graph of $\ln I_{DN}$ vs. $1/\cos \theta_z$, where $\theta_z$ is the solar zenith angle.

From data collected at the Air Wall Research Building (Ref. 20) on The Pennsylvania State University campus, several plots were developed and are shown in Fig. 6A. Also plotted on another graph (Fig. 6B) are the average summer data from "Air Wall," the standard curve recommended by Moon, and curves obtained by other researchers.

The curves plotted in Figs. 6A and B show differences in the $I_0$ values and the "a" values; however, the points plotted for each day fall in very straight lines indicating a consistency in the solar radiation behavior once a specific atmosphere is determined.

The above equation gives the direct solar energy for normal incidence. For the sun striking a surface at an angle, as shown in Fig. 4, the quantity of energy is expressed as the normal incidence value times the cosine of the angle of incidence to the surface (or $I_D = I_{DN} \cos \theta$).
For the horizontal surfaces this angle is $\theta_z$, the sun's zenith angle, as shown in Fig. 1. Therefore, $I_{DH} = I_{DN} \cos \theta_z = I_{DN} \sin \beta$.

For a vertical surface (see Fig. 4): $\cos \theta = \cos \beta \cos (\psi_z - \psi_N)$

therefore, $I_{DV} = I_{DN} \cos \beta \cos (\psi_z - \psi_N)$

PERIODIC HEAT FLOW

The following development is taken from Mackey and Wright (Refs. 13 and 14), but minor adjustments have been made in order to make computer programming more convenient.

Sol-air temperature $(t_e)$

$$t_e = t_a + bI/h_L$$

where, $I = I_D + I_d$

$b =$ solar absorptivity of surface

$h_L =$ value for outdoor air film coefficient of heat transfer = 4 Btu/hr, ft$^2$, °F

$t_a =$ °F of outdoor air

Exact solution for $t_o$

Assuming: Solar radiation and temperature of outdoor air are periodic

Indoor temperature is constant

Outdoor air film coefficient of heat transfer ($h_0$) is 4.0

Indoor air film coefficient of heat transfer ($h_L$) is 1.65

$$t_o = t_1 + \frac{.606 (t_m - t_1)}{1 + \frac{856 + L/k}{t_m}} + \sum_{n=1}^{\infty} \lambda_n t_n \cos (15nT - \alpha - \Phi_n)$$

where, $k =$ thermal conductivity (Btu/hr, ft, °F)

$L =$ material thickness (ft)

$t_m =$ mean daily sol-air temperature

$T =$ time (hours after noon)

$\Phi =$ harmonic lag angle

$a_n =$ harmonic phase angle

$\lambda_n =$ harmonic decrement factor

$t_n =$ harmonic temperature coefficient (*F)

Approximate solution for $t_o$

Steady-flow mean daily temperature of inside surface (for $h_o = 1.65$ and $h_L = 4.0$) is:

$$t_M = t_1 + \frac{.606 (t_m - t_1)}{1 + \frac{856 + L/k}{t_m}}$$

Steady-flow mean daily temperature of inside surface (for general case) is:

$$t_M = t_1 + \frac{1}{h_o} \sum_{n=1}^{\infty} \lambda_n t_n \cos (15nT - \alpha - \Phi_n)$$

Temperature inside surface of material $(t_o)$ at a time $(T + \Phi_1/15)$ hours after noon is:

$$t_0 = t_M + \lambda_1 (t^{**} - t_m)$$

where, $\Phi_1 =$ fundamental lag time

$\lambda_1 =$ fundamental decrement factor

$** =$ temperature at time $T + \Phi_1/15$

$* =$ temperature at time $T$

Recommended solution for $t_o$

The actual inside temperature of surface is:

$$t_o = t_M + \sum_{n=1}^{\infty} \lambda_n t_n \cos (15nT - a_n - \phi_n)$$

but temperature at time $T + \Phi_1/15$ is:

$$t_0 = t_M + \sum_{n=1}^{\infty} \lambda_n t_n \cos [15n(T + \phi_1) - a_n - \phi_n]$$

$$= t_M + \sum_{n=1}^{\infty} \lambda_n t_n \cos [15nT - a_n - (\phi_n - n\Phi_1)]$$

the first assumption is: $\phi_n \approx n\Phi_1$

the second assumption is: $(1 - \lambda_n) \approx (1 - \lambda_1)$

to obtain slightly more accurate results:

$$t_0 = t_M + \lambda_1 (t^{**} - t_m)$$

where, $\lambda_1 = (\lambda_1 + \lambda_2)/2 =$ average of fundamental and 2nd harmonic decrement factors
Calculation of decrement factor and lag angle

For fundamental harmonics:

\[ \lambda = \sqrt{\frac{2}{F^2 + G^2}} \]  
  decrement factor

\[ \phi = \tan^{-1} \left[ \frac{F - G}{F + G} \right] \]  
  lag angle

where,

\[ F = (\varphi_1 + 1) C_1 + (C_3 / \varphi_3) + 2 \varphi_1 \varphi_3 C_4 \]

\[ G = (\varphi_1 + 1) C_2 + (C_4 / \varphi_3) - 2 \varphi_1 \varphi_3 C_3 \]

\[ C_1 = \cos \varphi_2 \cosh \varphi_2 + \sin \varphi_2 \sinh \varphi_2 \]

\[ C_2 = \cos \varphi_2 \cosh \varphi_2 - \sin \varphi_2 \sinh \varphi_2 \]

\[ C_3 = \sin \varphi_2 \cosh \varphi_2 \]

\[ C_4 = \cos \varphi_2 \sinh \varphi_2 \]

\[ \varphi_1 = h_o / h_L \]

\[ \varphi_2 = s L \]

\[ \varphi_3 = k_s / h_o \]

\[ s = \sqrt{1.1309 \rho c / k} \]

for 2nd harmonic

\[ \varphi_{32} = \sqrt{2} \varphi_{31} \]

\[ \varphi_{22} = \sqrt{2} \varphi_{21} \]

For computer programming, the problem was written in terms of \((k \rho c)\) and \((L/k)\) as follows:

\[ \varphi_2 = (L/k) s \]

\[ \varphi_3 = s / h_o \]

\[ s = \sqrt{0.1309 (k \rho c)} \]

Nonhomogeneous walls and roofs (composite)

Reference 14 gives a method of establishing two thermal properties of composite construction to define the equivalent homogeneous construction—the thermal resistance \((L/k)_e\) and the quantity \((k \rho c)_e\). For multilayer construction the thermal resistance is given by:

\[ (L/k)_e = (L/k)_o + (L/k)_L + (L/k)_{m1} + (L/k)_{m2} + \ldots \]

The subscripts designate the following:

- \(e\) = equivalent
- \(o\) = inside layer
- \(L\) = outside layer
- \(ml\) = intermediate layer
- \(m2\) = second intermediate layer, etc.

The property \((k \rho c)_e\) of the equivalent homogeneous wall or roof is determined from the following empirical equation derived by Mackey and Wright (Refs. 13 and 14):

\[ (k \rho c)_e \approx \left( \frac{1.1 (L/k)_o (k \rho c)_o + 1.1 (L/k)_{m1} (k \rho c)_{m1} + \ldots}{(L/k)_e} \right) \]

\[ + \left( \frac{(L/k)_L - 0.1 (L/k)_o - 0.1 (L/k)_{m1} + \ldots}{(L/k)_e} \right) \]

Mackey and Wright also give a more exact method for calculating the decrement factors and lag angles, but it is very complicated and lengthy and requires a different calculational technique for each additional layer of material. The equation above, however, was tested by the computer (IBM 7074) using 42 different wall assemblies of two- and three-layer thicknesses, and results showed that the answers agreed very closely with those calculated by the very precise method by Mackey and Wright, except in the instances described below.

The author observed that the general equation for \(k \rho c\) was very exact, except for the following cases:

where the sum of the thermal resistances of the intermediate and inside layers was less than five times the thermal resistance of the outside layer, i.e.,

\[ \sum (L/k)_m + L/k_o \leq 5 (L/k)_L \]

and

where the opposite was also true, i.e.,

\[ L/k)_L \leq 5 \sum (L/k)_m + L/k_o \]
These two cases can be summarized in one expression:

\[ 2 \left( \frac{L}{k} \right)_L \leq \sum \left( \frac{L}{k} \right)_m + \frac{L}{k} \leq 5 \left( \frac{L}{k} \right)_L \]

Therefore, for all cases where the above expression is satisfied, the \( k/c \) equation will suffice and will produce fairly accurate results for decrement factors and lag angles.

In cases where the sum of the intermediate and inside thermal resistances fall outside the range specified in the above expression, another calculational procedure is used. This alternate procedure is a method recommended by Mackey and Wright and is outlined below:

- Find \( (L/k)_o \) in the regular manner by adding thermal resistances of each layer
- Compute the time lags for each of the layers as if each layer were used separately
- Total the time lags, and establish the result as the time lag of the composite wall
- Using the value of \( (L/k)_o \) and the time lag, find the required value of \( k/c \). (Using Mackey and Wright’s method, this can be done by reading values on a graph. Using a computer program, this value of \( k/c \) is established by a trial-and-error convergence technique.

The above four steps have been incorporated into a computer program to alter automatically the computation of the decrement factors and time lags as necessary. Forty-two different wall assemblages were computed and compared to Mackey and Wright’s exact solutions. The results were very close and are shown in Table 5.

### SUMMARY OF MODEL EQUATIONS

This section summarizes the equations described in the earlier sections of this report and those developed in this appendix. The intent of this summary is to organize the equations in a format that can be easily followed for the later development of computer programs which will use the equations. Complete equations are shown for calculating the instantaneous sensible heat gain from external conditions. Major sources of heat, such as people, lights, equipment, and appliances, are not described here since they can be included in the air-conditioning loads simply by adding them in the conventional way.

### Indoor Conditions

Summer - 75°F dry-bulb; RH = 50% in U.S.

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#### Table 5. Comparison Between Actual and Calculated Values for Wall Thermal Properties

<table>
<thead>
<tr>
<th>Wall Number</th>
<th>Fundamental Decrement Factors</th>
<th>Time Lags (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>calculated</td>
<td>actual</td>
</tr>
<tr>
<td>1</td>
<td>0.1610</td>
<td>0.1613</td>
</tr>
<tr>
<td>2</td>
<td>0.0984</td>
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<td>0.1613</td>
</tr>
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<td>0.0360</td>
<td>0.0361</td>
</tr>
<tr>
<td>7</td>
<td>0.0132</td>
<td>0.0133</td>
</tr>
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<tr>
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</tr>
</tbody>
</table>

1 Calculated by author  
2 Computed by Mackey and Wright (Ref. 14)
Outdoor conditions (temperatures and wind velocities)


Elevation effects on temperature values:
- Dry-bulb temperature changes -1°F per 200' increase in elevation
- Wet-bulb temperature changes -1°F per 500' increase in elevation

Ventilation Rate (cfm)

| Not smoking | 7 1/2 recommended | 5 minimum |
| Smoking     | 40 recommended    | 25 minimum |

Sun Position

- Solar declination angle (α)
  \[
  \sin \alpha = \sin 23.5° \cos \left( \frac{\pi d}{182.5} \right), \text{ or can be found in Table 2.}
  \]
- Altitude angle (β)
  \[
  \sin \beta = \sin \alpha \sin \phi - \cos \alpha \cos \phi \cos \left( \frac{\pi T}{12} \right)
  \]
- Azimuth angle (ψ)
  \[
  \cos \psi = \frac{\sin \phi \cos \alpha \cos \left( \frac{\pi T}{12} \right) + \cos \phi \sin \alpha}{\cos \beta}
  \]
- Time of sunrise (T_R)
  \[
  T_R = \left( \frac{12}{\pi} \right) \cos^{-1} \left[ \frac{\sin \alpha \sin \phi}{\cos \alpha \cos \phi} \right]
  \]
- Azimuth at sunrise (ψ_R)
  \[
  \cos \psi_R = \sin \alpha / \cos \phi
  \]
- Standard time (ST) vs. apparent solar time (AST)
  \[
  ST = AST - ET + 4 \ (\Delta \text{longitude})
  \]

Direct Solar Radiation

- Monthly average cloudiness index (K_T)
  \[
  K_T = \frac{H}{H_0}
  \]
- Extraterrestrial daily radiation on a horizontal surface (H_0)
  \[
  H_0 = \left( \frac{24}{\pi} \right) \times 442.4 \left[ \cos \phi \cos \alpha \sin \left( \frac{\pi T}{12} \right) + \left( \frac{\pi - \pi T}{12} \right) \sin \phi \sin \alpha \right]
  \]
- Cloudiness index (K_T)
  \[
  K_T = f(K_T), \text{ from Table 4 by selection of random numbers}
  \]
- Cloud cover (K_C)
  \[
  K_C = 1.6 (1.0 - K_T)
  \]
- Fraction of direct radiation from clear sky (K_D)
  \[
  K_D = 0.75 \quad \text{for } K_T > 0.80
  \]
  \[
  K_D = 1.415 K_T - 0.384 \quad \text{for } 0.38 \leq K_T \leq 0.80
  \]
- Fraction of direct radiation from cloudy or hazy sky (K_D')
  \[
  K_D' = 1.492 K_T - 0.492 \quad \text{for } 0.60 \leq K_T \leq 0.83
  \]
  \[
  K_D' = 0.935 K_T^2 - 1.0 \quad \text{for } K_T < 0.60
  \]
- Average daily intensity of direct solar radiation on a horizontal surface
  \[
  I_{DH} (\text{ave}) = K_D H_o / 2 (12 - T_R)
  \]
- Maximum solar altitude (β_max)
  \[
  \beta_{\max} = \left( \frac{\pi}{2} \right) - \phi + \alpha
  \]
- Atmospheric extinction coefficient (α)
  \[
  a = -\sin \beta_{\max} \log \left[ \frac{I_{DH} (\text{ave})}{(225 r \sin \beta_{\max})} \right]
  \]
- Direct normal solar radiation (I_DN)
  \[
  I_DN = 375 r e^{-a/\sin \beta}
  \]
- Direct radiation on horizontal (I_DH)
  \[
  I_{DH} = I_{DN} \sin \beta
  \]
- Direct radiation on vertical surface (I_DV)
  \[
  I_{DV} = I_{DN} \cos \beta \cos \left| \psi_z - \psi_N \right|
  \]
Diffuse Solar Radiation

- Transmission coefficient for direct radiation \( r_D \)
  \[
  r_D = \frac{I_{DN}}{I_{DN} + 442.4 r}
  \]
- Transmission coefficient for diffuse radiation for clear sky \( r_d \)
  \[
  r_d = 0.2710 - 0.2939 r_D
  \]
- Transmission coefficient for diffuse radiation for cloudy or hazy sky \( r_d' \)
  \[
  r_d = 0.33 (1 - r_D) \\
  \text{for } 0.4 \leq r_D < 0.4
  \]
  \[
  r_d = \sqrt{1.07 \log (r_D + 1)} - r_D \\
  \text{for } r_D < 0.4
  \]
- Proportionality constant (B)
  \[
  B = \left( \frac{r_d}{r_d'} \right) \sin \beta
  \]
- Diffuse radiation on horizontal surface \( I_{dh} \)
  \[
  I_{dh} = B I_{DN}
  \]
- Diffuse radiation on vertical surface \( I_d \)
  \[
  I_d = I_{dh} F (\cos \theta)
  \]

Periodic Heat Flow (Walls and Roofs)

- Total incident energy \( I \)
  \[
  I = I_D + I_d
  \]
- Sol-air temperature \( t_e \)
  \[
  t_e = t_a + \frac{b I}{h_L}
  \]
- Outdoor air film coefficient \( h_L \)
  \[
  h_L = 2.0 + 0.4 V_w
  \]

Heat Transfer Through Windows

- Total heat flow
  \[
  q = S \left[ F_d I_{DN} + F_d I_d + 0.27 (0.0236 F_d I_{DN} + 0.02 I_d) \right] + U (t_a - t_l)
  \]
- Fractional transmission of diffuse radiation \( F_d' \)
  \[
  F_d = 0.82
  \]
- Fractional transmission of direct radiation \( F_d' \)
  \[
  F_d = 0.865 e^{-0.019/\cos^2 \theta}
  \]

Total Energy Cost \( C_T \)

\[
C_T = \frac{Q}{T}
\]
APPENDIX 2. PROBLEM-ORIENTED COMPUTER LANGUAGE

The equations developed in this report will be used in future research as the central algorithms in a problem-oriented computer language. This language will permit two-way communication between the computer and the designer. Its purpose is to aid a designer in the selection of a wall system for a building by displaying the economic comparisons of various wall systems. The cost of each wall system will include the air-conditioning and maintenance costs of the wall for an entire year. Comparisons of wall systems will, therefore, show the true least cost wall for the building. Several wall systems should be evaluated by the designer in order that aesthetics be also considered in his selection.

The following is an example of a two-way communication between the designer and the computer in the selection of a wall system. Although the statements appear rather vague, they indicate the manner in which a designer would converse with the computer in the design and selection process. Communications would take place on a teletypewriter console such that both the sending message and receiving message are typed on the same paper. The statements labelled _C_ indicate the computer typed statements, and the statements, _D_, are those of the designer.

MODCON READY-ENTER REQUEST

Stage 1 - Tape U763

D Select wall systems for building
C Ready to select wall system for building
C Enter material no. 1
D None
C Enter all coordinates for material no. 1
(Coordinates for material no. 1 are entered through card reader)
C Enter material no. 2
D None
C Enter all coordinates for material no. 2
(Coordinates for material no. 2 are entered through card reader)
C Enter material no. 3
D Window
C Enter all coordinates for windows
(Coordinates for windows are entered through card reader)
C Enter material no. 4
D Door

C Enter material no. 5
C Enter all coordinates for doors
(Coordinates for doors are entered through card reader)
C Enter material no. 5
C Enter all materials in roof
D Steel joists, steel roof deck, insulation, built up roofing, bitumen and gravel
C Enter thermal properties under headings below:
  Thickness (in.)     Thermal conductivity     Vol. Specific Heat
C Steel joists
D None
C Steel roof deck
D None
C Insulation
D 2.0 0.025 4.0
C Built up roofing
D 1.0 0.10 6.0
C Bitumen and gravel
D 1.0 0.8 25.0
C What is absorptivity of outside surface for solar radiation?
D 0.2
C Are you finished with roof materials?
D Yes
C Enter all coordinates for roof
(Coordinates for roof are entered through card reader)
C Enter material no. 6
D Material entries finished
C Are you ready for wall system selection?
D Yes
C Do you have special list of wall systems for selection?
D No
Please eliminate (by number) the wall materials that should not be considered. See list: 1. brick, 2. rigid styrofoam insulation, 3. corkboard, etc.

Eliminate 4, 5, 7, 9, 12, 14, 18, 19, 20, 21, 25, 26, 28

Do you have any others to add?

No

Then the following wall materials will be considered in the wall system selection: Materials 1, 2, 3, 6, 8, 10, 11, 13, 15, 16, 17, 22, 23, 24, and 27

Verify - yes or no

Yes

Enter selection routine

Enter latitude and longitude

41° N, 178.4° W

Enter orientation - degree angle between normal to north wall and true north

-12 degrees

Enter weather tape

Read input tape

The following wall systems will be evaluated: Systems 1, 2, 3, 4, 5, 6, 8, and 9

Are there any more revisions?

Yes - eliminate system 5

Begin selection

Entering cost evaluation routine - stand by

Evaluation complete - elapsed time = 24 seconds

Results are shown below for the four systems of least total cost:

<table>
<thead>
<tr>
<th>System</th>
<th>Initial Cost</th>
<th>Total Cooling Cost</th>
<th>Total Heating Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Select the desirable system from the list above (by number)

Need to inspect elevations

Draw elevations

Elevations showing which of the four wall systems?