Abstract

We investigated very long but finite “ladder” circuits composed of alternating identical inductors and capacitors connected in series and parallel and derived an exact expression for the equivalent impedance of such circuits of arbitrary size. The remarkable simplicity of the impedance formula allowed us to directly obtain all resonance and anti-resonance frequencies. We tested our analytical results by constructing the corresponding circuits using the standard circuit simulation software (Multisim©). Specifically, we focused on circuits ranging from as few as one element to as many as a hundred elements. The equivalent impedances of these modeled circuits and the relevant voltage readings were in an excellent agreement with our theoretical calculations.

In addition, we resolved the well known paradoxical phenomenon arising from a naïve calculation of the equivalent impedance of an infinite LC-ladder; for driving frequencies below some critical value, the impedance of a purely reactive circuit seemed to acquire a non-zero active part. Using our formula, we demonstrated that there was no paradox and investigated the behavior of the equivalent impedance as the circuit size increased. We did so for various representative values of driving frequency and again found our theoretical predictions to be in agreement with the modeled circuits.

LC Ladder Paradox

The standard derivation of \( Z_{eq} \) is based on the recursion relation Eq.(1) that can be obtained by adding a new LC pair to an already existing circuit of \( k \) pairs. Traditionally the derivation proceeds as shown that \( Z_{k+1} = Z_{eq} + Z_{2} \) (Fig.1).

Using the rules for combining circuit elements in series and in parallel, we obtain the general equation for the equivalent impedance (Eq.2).

\[
Z_{eq} = \frac{Z_{1} + Z_{2} + Z_{2}}{2} \tag{2}
\]

Then, by using the known equations for impedances of an inductor (\( Z_{L} = \omega L \)) and capacitor (\( Z_{C} = \frac{1}{\omega C} \)), the general impedance formula becomes more specific for circuits composed of inductors and capacitors (Eq.3):

\[
Z_{eq} = \frac{1}{\frac{1}{\omega L} + \frac{1}{\omega C} + \omega^2} \tag{3}
\]

This result also used in Feynman’s Lectures [1] works elegantly for ladder circuits consisting of only resistors, but not so for LC ladder circuits because it gives rise to a questionable paradox. It gives a real a part for particular frequencies below its critical value \( \omega \) because inductors and capacitors only contain imaginary components.

Z Derivation

In order to derive the formula for the equivalent impedance of a ladder circuit composed of \( L \)- and C-poles, we can look at the behavior of a purely resistive ladder circuit of \( B = R_{1} + R_{2} + R_{3} \) (Fig.8). We can also derive a formula for the equivalent impedance of a ladder circuit composed of \( B = R_{1} + R_{2} + R_{3} \), as well as \( \theta \) and \( \omega \) as varies with \( n \). We can also derive a formula for the equivalent impedance of a ladder circuit composed of \( B = R_{1} + R_{2} + R_{3} \) as \( \omega \) varies with \( n \).

Theory

The impedance equation uses three values:

- \( \omega = \) Resonance frequency (based on L/C magnitude)
- \( k = \) Number of L/C pairs (based on length of circuit)
- \( \theta = \) an angle (based on frequency of circuit)

No effect to \( \varepsilon \) as \( \theta \) is not a factor. The equation provides a simple expression for the impedance that leads to inconsistencies. The equation offers another way to confirm our discoveries. A final analysis on the convergence of the impedance at both the critical and supercritical frequencies matched closely with the theoretical ones. A comparison of the measured voltage across the resistor and the theoretical values using Multisim’s “Single Frequency AC Analysis” simulation.

\[
V_{\text{Theory}} = \frac{ER}{\sqrt{R^2 + R^2}} \tag{7}
\]

As another way to confirm our results and ascertain that our circuit is properly set up, the voltage data across the resistor was taken and compared with calculated theoretical values using Eq.(7).

References

1. The Feynman Lectures on Physics. CalTech, 1964
8. The standard derivation of \( Z_{eq} \) is based on the recursion relation Eq.(1) that can be obtained by adding a new LC pair to an already existing circuit of \( k \) pairs. Traditionally the derivation proceeds as shown that \( Z_{k+1} = Z_{eq} + Z_{2} \) (Fig.1).

Data

The resonance frequency graph’s theoretical impedance values match with the analyzed data from Multisim. As predicted, it oscillates between \( -\omega \), \( \omega \), and \( \omega \).

Conclusion

After developing a simpler equation for the impedance of an infinite LC-ladder circuit, we analyzed very long ladder circuits composed of up to one hundred capacitors and inductors and found our data consistent with the theoretical values from our derived equation. As predicted, the circuit impedance oscillated between \( -\omega \), \( \omega \), and \( \omega \) at the natural frequency \( \omega_{res} \). As a side-effect, we also revealed that there is no paradox that arose in a previous calculation of the equivalent impedance commonly taught in physics classrooms. Specifically, it was the method of computing the impedance that led to inconsistencies. The method relied on the impedance’s convergence to a specific finite value as the circuit size went to infinity and worked well for purely reactive, inductive, or capacitive circuits, and even for LC-ladder circuits for supercritical frequencies. However, due to the aforementioned oscillations, the convergence is absent not only for the natural frequency, but also for any sub-critical frequency.

To test our equation further, in addition to the natural frequency, we analyzed the built circuit at three other key frequencies: irrational fraction of the resonance frequency and an exponential convergence for the super critical frequency. For the critical frequency, we graphed a log-log plot and for the super critical frequency, a semilog plot. As expected, the log-log plot of the critical frequency became linear, thus proving that the convergence follows a polynomial curve. However, the super critical log-log plot did not become linear as expected, and after more thinking, we decided that the convergence was not represented by one singular exponential function, but by multiple exponentials. Such a graph is more difficult to make linear, so we only fit the critical convergence.

Future Plans

Some future projects to consider are to investigate even longer ladders that can simulate the infinite characteristic of the ladder circuit even more, or test similar circuits in another software to see if it yields similar results. We can also take it a step further and build a physical model of our circuit. That in itself is an interesting and challenging approach which we considered doing initially, but decided to use software instead because of the efficiency of building the circuit and consistency of capacitors and inductors. The results obtained in this paper can help design multi-frequency LC-filters that permit/block signals at certain frequencies presented in Eq.(8). Another branch topic to look into is the analysis of an RLC coil network and complex periodic molecules, such as DNA etc.