Abstract

We investigated very long but finite "ladder" circuits composed of alternating identical inductors and capacitors connected in series and parallel and derived an exact expression for the equivalent impedance of such circuits of arbitrary size. The remarkable simplicity of the impedance formula allowed us to directly obtain all resonance and anti-resonance frequencies. We tested our analytical results by constructing the corresponding circuits using the standard circuit simulation software (Multisim™). Specifically, we focused on circuits ranging from as few as one element to as many as a hundred elements. The equivalent impedance of these modeled circuits and the relevant voltage readings were in an excellent agreement with our theoretical calculations.

In addition, we resolved the well known paradoxical phenomenon arising from a naïve calculation of the equivalent impedance of an infinite LC-ladder: for driving frequencies below some critical value, the impedance of a purely reactive circuit seemed to acquire a non-zero active part. Using our formula, we demonstrated that there was no paradox and investigated the behavior of the equivalent impedance as the circuit size increased. We did so for various representative values of driving frequency and found our theoretical predictions to be in agreement with the modeled circuits.

In this investigation we focused on circuits ranging from as few as one element to as many as a hundred elements. The equivalent impedance of these modeled circuits and the relevant voltage readings were in an excellent agreement with our theoretical calculations.

Two options for testing:
1. Physical build
2. Computer simulation

For testing purposes, we constructed 100 circuits with 1 to 100 total units (Fig-2).

The resonance frequency graph's theoretical impedance values match with the analyzed data from Multisim. As predicted, it oscillates between 0, ±ω0, and +∞.

The impedance equation uses three values:
- Z0 = Natural frequency = \frac{1}{ω0}
- k = Number of LC pairs (based on length of circuit)
- \theta = \alpha \frac{Z}{Z_{eq}} (\text{ω} \text{ driving frequency})

The analysis:
- Four key frequencies of interest were chosen and their respective resistor added in series with the ladder portion of the circuit to allow Multisim to display the voltage, which unlike the impedance, is a finite value and can be displayed more consistently.
- Multiplied of very high resistance added in parallel with the resistor to measure the voltage across the resistor
- "Probes," added

The equation was used to predict a general pattern for the impedance at the natural frequency.

Convergence Graph for Critical Frequency

After finding the circuit impedance at both the critical and supercritical frequencies, the two graphs converged to the impedance calculated using Eq-3. As a further analysis, we estimated the rate of convergence by finding an equation to represent its type. We estimated the frequency at which the circuit impedance converges to its theoretical value (ω0). As a side-effect, we also realized that there is no paradox that arose in a previous calculation of the equivalent impedance. Specifically, it was the method of computing the impedance that led to inconsistencies. The method relied on the circuit's impedance to a specific finite value as the circuit size went to infinity and worked well for purely resistive, inductive, capacitive, and even for LC-ladder circuits for supercritical frequencies. However, due to the aforementioned oscillations, the convergence is absent not only for the natural frequency, but also for any sub-critical frequency.

To test our equation further, in addition to the natural frequency, we analyzed the built circuits at three other critical frequencies: irrational fraction of the resonance frequency, the critical frequency, and a super critical frequency, and compared the analyzed impedance values with the theoretical ones from our equation. As shown in (Fig-4), both the values and behavior of the impedances matched closely with the theoretical ones. A comparison of the measured voltage across the added resistor and its theoretical value also yielded similar results and offered another way to confirm our discoveries. A final analysis on the convergence of the impedance at both the critical and supercritical frequencies revealed that the circuit impedance at a super critical frequency converged to its limiting value exponentially fast.

Future Plans

Some future projects to consider are to investigate even longer circuits that can initiate the infinite characteristic of the ladder circuit even more, or test similar circuits in another software to see if it yields similar results. We can also take it a step further and build a physical model of our circuit. That in itself is an interesting and challenging approach which we considered doing initially, but decided to use software instead because of the efficiency of building the circuit and consistency of capacitors and inductors. The results obtained in this paper can help design multi-frequency LC-filters that permit/block signals at certain frequencies presented in Eq 6. Another branch topic to look into is the analysis of an RLC colinear network and complex periodic molecules, such as DNA etc.