DESCRIPTION OF COSTANDARD MODULES OF SCHUR SUPERA LGE BRA $S(3|1)$

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Abstract. Let $S(m|n)$ be a Schur superalgebra over a base field $K$ of any characteristics different from 2. The characters of simple $S(m|n)$-modules and composition factors of costandard modules $\nabla(\lambda)$ for restricted weights $\lambda$ were determined for the Schur superalgebra $S(2|1)$ in [6] and for Schur superalgebra $S(2|2)$ in [4] and [5]. The goal of this paper is to obtain corresponding results for Schur superalgebra $S(3|1)$. Additionally, we determine the character of $\nabla(\lambda)$ for all Schur superalgebras $S(m|1)$, where $m \geq 1$.

Introduction

Throughout the paper we shall work over an algebraically closed field $K$ of characteristic $p = 0$ or $p > 2$ and use the basic terminology of bialgebras $A(m|n)$, general linear supergroup $GL(m|n)$, Schur superalgebras $S(m|n)$ and superderivations $i_j D$ from papers [14, 10]. All modules considered in this paper will be left modules and all superderivations will be right superderivations.

In this manuscript we shall mostly work with $G = GL(3|1)$ and $S(3|1)$ and we describe them in a down-to-earth fashion in Section 1. Throughout the paper $\lambda = (\lambda_1, \lambda_2, \lambda_3 | \lambda_4)$ will denote an integral dominant weight of $G$, that is $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Let $G_{ev} \cong GL(3) \times GL(1)$ be an even supersubgroup of $G$ and $S = SL(3) \times SL(1) \cong SL(3)$ be a corresponding even supergroup. There is a natural correspondence between weights for $G_{ev}$-modules and $S$-modules given by $\lambda = (\lambda_1, \lambda_2, \lambda_3 | \lambda_4) \mapsto (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3)$. It is easy to recover the $G_{ev}$-structure of a module from its $S$-structure.

Let $B$ be the lower standard Borel subsupergroup of $G$ and $K_\lambda$ be the one-dimensional (even) $B$-supersubmodule corresponding to the weight $\lambda$. Following [14], we denote by $H^0_G(\lambda)$ the $G$-module $H^0(G/B, K_\lambda)$ which is isomorphic to the induced module $\text{ind}^G_B(K_\lambda)$, and by $H^0_{G_{ev}}(\lambda)$ the induced $G_{ev}$ -module corresponding to the weight $\lambda$.

The $G$-supermodule $H^0_G(\lambda)$ can be described explicitly using the isomorphism $\phi : H^0_{G_{ev}}(\lambda) \otimes S(X_{12}) \rightarrow H^0_G(\lambda)$ defined in Lemma 5.2 and on p.163 of [14]. In Section 1 we determine important formulas that shall be used repeatedly, in particular we compute the action of superderivations $i_j D$ on $H^0_{G_{ev}}(\lambda)$ and on elements $\phi(X_{12})$. Furthermore, we introduce the concept of a typicality of the weight $\lambda$.

Our approach to the description of simple $S(3|1)$-modules consists of two steps. In the first step we determine the $S$-module structure of $H^0_G(\lambda)$. For this step, we divide $H^0_G(\lambda)$ into natural segments, called floors, that are subsequently identified.
as tensor products of \( H_{G,w}^0(\lambda) \) with exterior powers of a certain three-dimensional module \( Y \). In the second step, we study the image of these floors under endomorphisms of \( S \)-modules that correspond to composition of various superderivations \( \delta_j D \). We determine the \( S \)-module structures of these images that together combine to form the simple \( S(3|1) \)-module \( L_{S(3|1)}(\lambda) \).

This process is easier in the case of characteristic zero since the category of \( S(3|1) \)-modules are semisimple. Both steps of this process are completed in Section 2.

The remainder of the paper deals with the case when the characteristic \( p \) of the ground field \( K \) is bigger than two. In order to describe all simple \( S(3|1) \)-modules, using the Steinberg Theorem (see Theorem 4.4 of [9]), it is enough to determine the structure of the simple \( S(3|1) \)-module \( L_{S(3|1)}(\lambda) \) for \( \lambda \) restricted.

In Section 3 we work with certain tensor products and determine the \( S \)-module structure of the floors of \( H_{G}^0(\lambda) \) for restricted \( \lambda \). Special care is taken in the cases when \( \lambda \) lies on the wall of an alcove of an affine Weyl group, that is when \( \lambda_1 - \lambda_2 \) or \( \lambda_2 - \lambda_3 \) is equal to \( p - 1 \), or \( \lambda_1 - \lambda_3 = p - 2 \).

In Section 4, we give an explicit description of images of these floors under applications of consecutive superderivations, and consequently, determine the \( S \)-module structure of the simple \( S(3|1) \)-modules for restricted \( \lambda \).

In Section 5 we determine the character of all simple \( S(3|1) \)-modules.

In Section 6 we compute all simple composition factors in the filtration of costandard modules \( \nabla(\lambda) \), that is the polynomial part of \( H_{G}^0(\lambda) \), for every restricted hook \( \lambda \) such that \( \lambda_3 > 0 \). Thus we determine all decomposition numbers in the process of modular reduction of simple \( S(3|1) \)-modules with such restricted highest weights.

In Section 7 we describe the character of \( \nabla(\lambda) \) and determine the simple composition factors of \( \nabla(\lambda) \) for restricted weights \( \lambda \) such that \( \lambda_3 = 0 \).

In Section 8 we work over the Schur superalgebra \( S(m|1) \), where \( m \geq 1 \) and determine the character of its costandard modules \( \nabla(\lambda) \).

In Section 9 we clarify and correct some issues in [6] regarding the description of costandard modules for Schur superalgebra \( S(2|1) \).

Finally, we would like to remark that all results in this paper are derived for modules over the Schur superalgebra \( S(m|n) \) which correspond to polynomial \( GL(m|n) \)-modules. Unlike in the classical case of the general linear group \( GL(n) \), we do not obtain a description of all modules over the \( GL(m|n) \) since the trick of multiplication by the determinant does not work in the supercase. This is due to the fact that the Berezinian element and the corresponding one-dimensional module are not polynomial.

1. Basic formulas and notation

For the description of the simple and costandard modules for the general linear group \( GL(3) \) and the Schur algebra \( S(3) \) we refer the reader to [2, 7, 11, 12].

1.1. Basic formulas for \( S(3|1) \). To define the general linear group \( G = GL(3|1) \) explicitly, we start with a commutative superalgebra \( A = A(3|1) \) freely generated over \( K \) by elements \( c_{ij} \) for \( 1 \leq i, j \leq 4 \), where \( c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}, c_{31}, c_{32}, c_{33} \) and \( c_{44} \) are even and \( c_{14}, c_{24}, c_{34} \) are odd. Denote \( \mathfrak{e} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \).
The superalgebra $A$ has a structure of a bialgebra given by comultiplication $\delta : A \to A \otimes A$ defined as $\delta(c_{ij}) = \sum_k c_{ik} \otimes c_{kj}$. The localization of $A(3|1)$ by elements $D = \det(\mathfrak{C})$ and $c_{44}$ is the coordinate superalgebra of the general linear supergroup $G$. The general linear supergroup $G$ is a group functor from the category $SAlg_K$ of commutative superalgebras over $K$ to the category of groups, represented by its coordinate ring $K[G]$, that is $G(A) = Hom_{SAlg_K}(K[G], A)$ for $A \in SAlg_K$. Here, for $g \in G(A)$ and $f \in K[G]$ we define $f(g) = g(f)$. The dual of the coalgebra $A$ is the Schur superalgebra $S(3|1)$ which corresponds to polynomial representations of $G$.

Assume now that the characteristic $p = 0$. We will derive basic formulas in this case and then indicate how to modify them for the case $p > 2$ at the beginning of Section 3.

Denote by $Adj(\mathfrak{C}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$ the adjoint of the matrix $\mathfrak{C}$, that is

$A_{11} = -c_{23}c_{32} + c_{22}c_{33}, \quad A_{12} = c_{13}c_{32} - c_{12}c_{33}, \quad A_{13} = -c_{13}c_{22} + c_{12}c_{23},$

$A_{21} = c_{23}c_{31} - c_{21}c_{33}, \quad A_{22} = -c_{13}c_{31} + c_{11}c_{33}, \quad A_{23} = c_{13}c_{21} - c_{11}c_{23},$

$A_{31} = -c_{22}c_{31} + c_{21}c_{32}, \quad A_{32} = c_{12}c_{31} - c_{11}c_{32}, \quad A_{33} = -c_{12}c_{21} + c_{11}c_{22}.$

The induced $G_{ev}$-module $H^0_{G_{ev}}(\lambda)$, denoted by $V$, can identified with the subspace of superalgebra $B = K[c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}, c_{31}, c_{32}, c_{33}, c_{44}]$, generated by polynomials

$\nu_{a,b,c,d} = D^{\lambda_2}d_2^{\lambda_3}d_3^{\lambda_4}c_{11}^{a}c_{12}^{b}c_{13}^{d}c_{44}^{c},$

where $D = \det(\mathfrak{C})$, $d_{12} = A_{33}$, $d_{13} = -A_{23}$, $d_{23} = A_{13}$, $0 \leq a, b, c, d$, and $a + b \leq \lambda_2$, $c + d \leq \lambda_1 - \lambda_2 - \lambda_3$.

The set of the polynomials as above form a basis of $H^0_{G_{ev}}(\lambda)$ if we require additionally that $c = 0$ whenever $a + b < \lambda_2 - \lambda_3$. This condition ensures that the corresponding polynomial is a semistandard bideterminant.

To facilitate a conversion to semistandard bideterminants we will need the formula

$\nu_{a,b,c,d} = \nu_{a,b+1,c-1,d+1} - \nu_{a+1,b,c-1,d}$ for $c > 0$.

(1)

The $G$-supermodule $H^0_{G}(\lambda)$ can be described using an explicit isomorphism $\phi : H^0_{G_{ev}}(\lambda) \otimes K[c_{14}, c_{24}, c_{34}] \to H^0_{G}(\lambda)$ defined in Lemma 5.2 and on p.163 of [10].

Isomorphism $\phi$ is given by

$\phi(D) = D, \quad \phi(d_{12}) = d_{12}, \quad \phi(d_{13}) = d_{13}, \quad \phi(d_{23}) = d_{23},$

$\phi(c_{11}) = c_{11}, \quad \phi(c_{12}) = c_{12}, \quad \phi(c_{13}) = c_{13},$

$\phi(c_{14}) = A_{11}c_{14} + A_{12}c_{24} + A_{13}c_{34} = y_{14},$

$\phi(c_{24}) = A_{21}c_{14} + A_{22}c_{24} + A_{23}c_{34} = y_{24},$

$\phi(c_{34}) = A_{31}c_{14} + A_{32}c_{24} + A_{33}c_{34} = y_{34},$

$\phi(c_{44}) = c_{44} - c_{14}y_{14} - c_{24}y_{24} - c_{34}y_{34} = z_4.$

Then $G$-supermodule $H^0_{G}(\lambda)$ has a basis

$\omega(a,b,c,d,\delta_{14},\delta_{24},\delta_{34}) = D^{\lambda_2}d_2^{\lambda_3}d_3^{\lambda_4}c_{11}^{a}c_{12}^{b}c_{13}^{d}c_{44}^{c} y_{14}^{\delta_{14}} y_{24}^{\delta_{24}} y_{34}^{\delta_{34}},$

where $a, b, c, d$ as before and such that $c = 0$ whenever $a + b < \lambda_2 - \lambda_3$, and $\delta_{14}, \delta_{24}, \delta_{34} \in \{0, 1\}$.
The weight of \( w(a, b, c, d, \delta_{14}, \delta_{24}, \delta_{34}) \) is

\[
(\lambda_3 + a + b + c - \delta_{14}, \lambda_2 - b + d - \delta_{24}, \lambda_1 - a - c - d - \delta_{34} | \lambda_4 + \delta_{14} + \delta_{24} + \delta_{34}).
\]

We shall write \( v_{a, b, c, d} \) for \( w(a, b, c, d, 0, 0, 0) \).

Since elements \( y_{14}, y_{24} \) and \( y_{34} \) are odd, \( (c_{14}y_{14} + c_{42}y_{24} + c_{43}y_{34})^k \neq 0 \) only if \( k \leq 3 \). Moreover, \( (c_{41}y_{14} + c_{42}y_{24} + c_{43}y_{34})^3 = 6c_{41}y_{14}c_{42}y_{24}c_{43}y_{34} \) and \( (c_{41}y_{14} + c_{42}y_{24} + c_{43}y_{34})^2 = 2c_{41}y_{14}c_{42}y_{24} + 2c_{41}y_{14}c_{43}y_{34} + 2c_{42}y_{24}c_{43}y_{34} \). Therefore we have eight types of basis elements

\[
w(a, b, c, d, \delta_{14}, \delta_{24}, \delta_{34}) = D^{\lambda_3}d_1^{\delta_{14}}d_2^{\delta_{24}}d_3^{\delta_{34}} - \lambda_3 - a - b - c - d \cdot c_{11}d_1 c_{12}d_2 c_{13}d_3 c_{14}d_4 - \lambda_2 - c - d \cdot c_{14}d_4^{-1} (c_{14}y_{14} + c_{42}y_{24} + c_{43}y_{34})
\]

\[
- \lambda_4 (\lambda_4 - 1) (c_{41}y_{14}c_{42}y_{24} + c_{41}y_{14}c_{43}y_{34} + c_{42}y_{24}c_{43}y_{34})
\]

\[
- \lambda_4 (\lambda_4 - 1) (\lambda_4 - 2) c_{41}y_{14}c_{42}y_{24}c_{43}y_{34}
\]

\[
y_{14}y_{24} y_{34}^{\delta_{14}} y_{24}^{\delta_{24}} y_{34}^{\delta_{34}}
\]

depending on the values of \( \delta_{14}, \delta_{24}, \delta_{34} \).

Observe that

\[
y_{14}y_{24} = \frac{c_{33}c_{14}c_{24} - c_{23}c_{14}c_{34} + c_{13}c_{24}c_{34}}{D}
\]

\[
y_{14}y_{34} = \frac{-c_{32}c_{14}c_{24} + c_{22}c_{14}c_{34} - c_{12}c_{24}c_{34}}{D}
\]

\[
y_{24}y_{34} = \frac{c_{31}c_{14}c_{24} - c_{21}c_{14}c_{34} + c_{11}c_{24}c_{34}}{D}
\]

\[
y_{14}y_{24}y_{34} = \frac{c_{14}c_{24}c_{34}}{D}.
\]

Therefore \( D y_{14}, D y_{24}, D y_{34}, D y_{14}y_{24}, D y_{14}y_{34}, D y_{24}y_{34}, D y_{14}y_{24}y_{34} \) and \( D y_{14}y_{24}y_{34} \) are all polynomial elements.

Therefore, if \( \lambda_3 > 0 \), then all \( w(a, b, c, d, \delta_{14}, \delta_{24}, \delta_{34}) \) are polynomials and \( \nabla(\lambda) = H_G^{(0)}(\lambda) \).

Define the right superderivations \( ij \cdot D \) of the coalgebra \( A(3|1) \) by \( (c_{ki})_{ij} \cdot D = c_{kj} \) and \( (c_{ki})_{ij} \cdot D = 0 \) for \( i \neq j \).

At this time we would like to remark that the definition of the superderivation \( D_{ij} \) in [6] was stated incorrectly. The correct definition is as above and in the terminology of [6] it should read \( (c_{ki})_{ij}^{D} = \delta_{ii}c_{kj} \).

We now state the formulas for actions of superderivations \( ij \cdot D \) on generating elements.

**Lemma 1.1.** The actions of superderivations \( 12D, 13D, 14D, 23D, 24D, 32D, 34D, 43D \) on elements \( D, d_{12}, d_{13}, d_{23}, c_{11}, c_{12}, c_{13}, y_{14}, y_{24}, y_{34} \) and \( z_4 \) is given in the following tables.
Lemma 1.2.

Proof. It is straightforward computation using the properties \((c_{kl})_{ij} D = \delta_{ti} c_{kj}\), where \(\delta_{ti}\) is the Kronecker delta, and \((ab)_{ij} D = (-1)^{|i| D |b|} (a)_{ij} D b + a(b)_{ij} D\), where the symbol | | denotes the parity.

Using these formulae we obtain the following Lemma.

**Lemma 1.2.**

\[
\begin{align*}
(v_{a,b,c,d})_{12} D &= b v_{a,b-1,c,d} + c v_{a,b,c-1,d+1} \\
&= (b + c) v_{a,b,c-1,d+1} - b v_{a+1,b-1,c-1,d} & \text{if } c > 0 \\
&= b v_{a,b-1,c,d} & \text{if } c = 0,
\end{align*}
\]

\[
\begin{align*}
(v_{a,b,c,d})_{23} D &= a v_{a-1,b+1,c,d} + d v_{a,b,c,d-1}, \\
(v_{a,b,c,d})_{13} D &= -a v_{a-1,b,c,d} + c v_{a,b,c-1,d} \\
&= (a + c) v_{a,b,c-1,d} - a v_{a-1,b+1,c-1,d+1} & \text{if } c > 0 \\
&= -a v_{a-1,b,c,d} & \text{if } c = 0,
\end{align*}
\]

\[
\begin{align*}
(v_{a,b,c,d})_{24} D &= (\lambda_2 - \lambda_3 - a - b) v_{a,b+1,c,d} + d v_{a,b,c+1,d-1}, \\
(v_{a,b,c,d})_{32} D &= b v_{a+1,b-1,c,d} + (\lambda_1 - \lambda_2 - c - d) v_{a,b,c,d+1}, \\
(v_{a,b,c,d})_{31} D &= -(\lambda_2 - \lambda_3 - a - b) v_{a+1,b,c,d} + (\lambda_1 - \lambda_2 - c - d) v_{a,b,c+1,d}.
\end{align*}
\]
Finally, if circumstances.

Moreover, if $v$ is called \((14,24,34)\)-atypical if

\[
(\nu_{a,b,c, d})_{14} D = (\lambda_3 + \lambda_4 + a + b + c) v_{a,b,c, d} y_{14} + b v_{a,b-1,c, d} y_{24} + c v_{a,b,c-1,d+1} y_{24} \\
- a v_{a-1,b,c,d} y_{34} + c v_{a,b,c-1,d+1} y_{34} \\
= (\lambda_3 + \lambda_4 + a + b + c) v_{a,b,c, d} y_{14} + (v_{a,b,c, d})_{12} D y_{24} + (v_{a,b,c, d})_{13} D y_{34},
\]

\[
(\nu_{a,b,c, d})_{24} D = d v_{a,b,c+1,d-1} y_{14} + (\lambda_2 - \lambda_3 - a - b) v_{a,b+1,c,d} y_{14} + (\lambda_2 + \lambda_4 - b + d) v_{a,b,c,d} y_{24} \\
+ a v_{a-1,b+1,c,d} y_{34} + d v_{a,b,c,d-1} y_{34} \\
= (v_{a,b,c, d})_{21} D y_{14} + (\lambda_2 + \lambda_4 - b + d) v_{a,b,c,d} y_{24} + (v_{a,b,c, d})_{23} D y_{34},
\]

\[
(\nu_{a,b,c, d})_{34} D = - (\lambda_2 - \lambda_3 - a - b) v_{a+1,b,c,d} y_{14} + (\lambda_1 - \lambda_2 - c - d) v_{a,b,c+1,d} y_{14} + b v_{a+1,b-1,c,d} y_{24} \\
+ (\lambda_1 + \lambda_4 - a - c - d) v_{a,b,c,d} y_{34} \\
= (v_{a,b,c, d})_{31} D y_{14} + (v_{a,b,c, d})_{32} D y_{24} + (\lambda_1 + \lambda_4 - a - c - d) v_{a,b,c,d} y_{34}
\]

\[
(\nu_{a,b,c, d})_{41} D = (v_{a,b,c, d})_{42} D = (v_{a,b,c, d})_{43} D = 0.
\]

Moreover, if $c > 0$, then

\[
(\nu_{a,b,c, d})_{12} D = (b + c) v_{a,b,c-1,d+1} - b v_{a+1,b-1,c-1,d}
\]

and

\[
(\nu_{a,b,c, d})_{13} D = (a + c) v_{a,b,c-1,d} - a v_{a-1,b+1,c-1,d+1}.
\]

Finally,

\[
(\nu_{a,b,0,d})_{12} D = -a v_{a-1,b,0,d} \text{ and } (\nu_{a,b,0,d})_{13} D = b v_{a,b-1,0,d}.
\]

**Proof.** The proof uses repeated application of Lemma 1.1 and Equation 1. \qed

1.2. **Further notation.** The simple $S$-module of the highest weight $\mu$ and of the highest vector $w$ shall be denoted either by $L(\mu)$ or by $L(w)$ depending on the circumstances.

The $S$-module generated by vector $v$ will be denoted by $\langle v \rangle$.

A dominant $G$-weight $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ will be called restricted if $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3 < p$.

Denote $\lambda_1 - \lambda_2 = C$ and $\lambda_2 - \lambda_3 = A$. Further denote $\omega_{12} = \lambda_1 - \lambda_2, \omega_{23} = \lambda_2 - \lambda_3, \omega_{14} = \lambda_1 + \lambda_4 + 2, \omega_{24} = \lambda_2 + \lambda_4 + 1, \omega_{34} = \lambda_3 + \lambda_4$.

If $p = 0$, then we shall write $\delta_{ij} = 0$ if $\omega_{ij} = 0$ and $\delta_{ij} = 1$ otherwise. If $p > 2$, then we denote $\delta_{ij} = 0$ if $\omega_{ij} \equiv 0 \pmod{p}$ and $\delta_{ij} = 1$ otherwise.

The following definition is motivated by the definition of a typical and an atypical weight appearing in the work of Kac [8] on Lie superalgebras in characteristic zero.

**Definition 1.3.** A dominant weight $\lambda$ is called typical if $\omega_{14} \omega_{23} \omega_{34} \not\equiv 0 \pmod{p}$, $\lambda$ is called 14-atypical if $\omega_{14} \equiv 0 \pmod{p}$ but $\omega_{23} \omega_{34} \not\equiv 0 \pmod{p}$, $\lambda$ is called 24-atypical if $\omega_{24} \equiv 0 \pmod{p}$ but $\omega_{13} \omega_{34} \not\equiv 0 \pmod{p}$, $\lambda$ is called 34-atypical if $\omega_{34} \equiv 0 \pmod{p}$ but $\omega_{14} \omega_{24} \not\equiv 0 \pmod{p}$, $\lambda$ is called (14,24)-atypical if $\omega_{14} \equiv 0 \pmod{p}$, $\omega_{24} \equiv 0 \pmod{p}$ but $\omega_{34} \not\equiv 0 \pmod{p}$, $\lambda$ is called (14,34)-atypical if $\omega_{14} \equiv 0 \pmod{p}$, $\omega_{34} \equiv 0 \pmod{p}$ but $\omega_{24} \not\equiv 0 \pmod{p}$, $\lambda$ is called (24,34)-atypical if $\omega_{24} \equiv 0 \pmod{p}$, $\omega_{34} \equiv 0 \pmod{p}$ but $\omega_{14} \not\equiv 0 \pmod{p}$, $\lambda$ is called (14,24,34)-atypical if $\omega_{14} \equiv 0 \pmod{p}$, $\omega_{24} \equiv 0 \pmod{p}$ and $\omega_{34} \equiv 0 \pmod{p}$.
Observe the following. If \( \lambda \) is 14-atypical, then \( C \neq p - 1 \). If \( \lambda \) is 24-atypical, then \( A \neq p - 1 \) and \( C \neq p - 1 \). If \( \lambda \) is 34-atypical, then \( A \neq p - 1 \). If \( \lambda \) is (14,24)-atypical, then \( A \neq p - 1 \) and \( C = p - 1 \). If \( \lambda \) is (24,34)-atypical, then \( A = p - 1 \) and \( C \neq p - 1 \). If \( \lambda \) is (14,34)-atypical, then \( A + C = p - 2 \). If \( \lambda \) is (12,24,34)-atypical, then \( A = p - 1 \) and \( C = p - 1 \).

In what follows we shall divide all weights \( \mu = (\mu_1, \mu_2, \mu_3) \) of \( H^0_G(\lambda) \) into four floors. For \( i = 0, 1, 2, 3 \), the \( i \)-th floor \( F_i(\lambda) \) of \( H^0_G(\lambda) \) is spanned by weights \( \mu \) for which \( \mu_1 + \mu_2 + \mu_3 = \lambda_1 + \lambda_2 + \lambda_3 - i \). Clearly each \( F_i(\lambda) \) is a \( S \)-module. Denote by \( Y \) a three-dimensional \( S \)-module spanned by elements \( y_{14}, y_{24} \) and \( y_{34} \). Then \( F_0(\lambda) = V, F_1(\lambda) = V \oplus Y, F_2(\lambda) = V \oplus (Y \wedge Y) \) and \( F_3(\lambda) = V \otimes (Y \wedge Y \wedge Y) \).

Assume that \( A, B > 0 \).

The elements \( (v)_{14} D, (v)_{12} D_{24} D, (v)_{12} D_{23} D_{34} D \) and \( (v)_{23} D_{12} D_{34} D \) of \( F_1(\lambda) \) are linear combinations of elements

\[
B_1 = \{ \langle v \rangle_{14} D y_{12}, (v)_{12} D_{23} D y_{34}, (v)_{23} D_{12} D y_{34} \}.
\]

The coefficient matrix is

\[
\begin{pmatrix}
\lambda_1 + \lambda_4 & 1 & 1 & -1 \\
\lambda_1 - \lambda_2 & \lambda_2 + \lambda_4 + 1 & 1 & 0 \\
\lambda_1 - \lambda_2 & \lambda_2 - \lambda_3 + 1 & \lambda_3 + \lambda_4 + 1 & 0 \\
-(\lambda_2 - \lambda_3) & \lambda_2 - \lambda_3 & 0 & \lambda_3 + \lambda_4 + 1
\end{pmatrix}
\]

and its determinant equals \( \omega_{14} \omega_{24} \omega_{34} \).

The elements \( (v)_{14} D_{24} D, (v)_{23} D_{34} D_{14} D, (v)_{12} D_{23} D_{34} D_{24} D \) and \( (v)_{23} D_{12} D_{34} D_{24} D \) of \( F_2(\lambda) \) are linear combinations of elements

\[
B_2 = \{ \langle v \rangle_{14} y_{24}, (v)_{23} D y_{14} y_{34}, (v)_{12} D_{23} y_{14} y_{34}, (v)_{23} D_{12} D y_{24} y_{34} \}.
\]

The coefficient matrix is

\[
\begin{pmatrix}
(\lambda_1 + \lambda_4 + 1)(\lambda_2 + \lambda_4) & \lambda_1 + \lambda_4 + 1 & -\lambda_2 + \lambda_4 + 1 & \lambda_2 + \lambda_4 + 1 \\
-(\lambda_2 - \lambda_3) (\lambda_1 + \lambda_4 + 1) & -\lambda_2 + \lambda_4 + 1 & \lambda_1 - \lambda_3 & -\lambda_2 + \lambda_4 + 1 \\
(\lambda_1 - \lambda_2)(\lambda_3 + \lambda_4) & -(\lambda_1 - \lambda_2)(\lambda_3 + \lambda_4) & -(\lambda_3 + \lambda_4)(\lambda_1 - \lambda_2 + 1) & \lambda_2 - \lambda_3 \\
-(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_4 + 1) & -(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_4 + 1)(\lambda_1 - \lambda_2 + 1) & \lambda_2 - \lambda_3 & -(\lambda_3 + \lambda_4)(\lambda_1 - \lambda_2 + 1)
\end{pmatrix}
\]

and its determinant equals \( -\omega_{14}^2 \omega_{24} \omega_{34} \).

Finally, on the third floor we have

\[
(v)_{14} D_{24} D_{34} D = \omega_{14} \omega_{24} \omega_{34} \langle v \rangle_{14} y_{24} y_{34}.
\]

When considering structure of \( S \)-modules, it is convenient to view them as \( SL(3) \)-modules only. We will, by abuse of notation, write \( \mu = (\mu_1 - \mu_2, \mu_2 - \mu_3) \) instead of \( \mu = (\mu_1, \mu_2, \mu_3) \) and use the corresponding notation from \( SL(3) \).

Denote by \( S(\mu) = S(\mu, x_1, x_2, x_3) \) the Schur function corresponding to \( \mu \) and by \( \chi(\mu) = \chi(x_1, x_2, x_3) \) the character of the simple \( S \)-module \( L(\lambda) \). Because we will refer to \( SL(3) \)-notation frequently, it will be more convenient to use the functions \( P(\mu_1, \mu_2, \mu_3) = S_{\mu_1 + \mu_2 + \mu_3, \mu_2 + \mu_3, \mu_3}(x_1, x_2, x_3) \) instead of \( S(\mu) \).

In what follows we will use the following notations for some particular weights. Denote \( \alpha_{12} = (-1, 0, 0), \alpha_{23} = (0, -1, 1), \alpha_{14} = (-1, -1, 0), \gamma_{14} = (-1, 0, 0), \gamma_{24} = (0, -1, 0), \gamma_{34} = (0, 0, 0), \Lambda_{14} = \lambda + \gamma_{14}, \Lambda_{24} = \lambda + \gamma_{24}, \Lambda_{34} = \lambda + \gamma_{34}, \Lambda_{14, 24} = \lambda + \gamma_{14} + \gamma_{24}, \Lambda_{14, 34} = \lambda + \gamma_{14} + \gamma_{34} \) and \( \Lambda_{14, 24, 34} = \lambda + \gamma_{14} + \gamma_{24} + \gamma_{34} \).

Denote also \( \Lambda_{34} = \lambda_{24} + (A + C - p + 3)(\alpha_{12} + \alpha_{23}), \Lambda_{24} = \lambda_{24} + (A + C - p + 2)(\alpha_{12} + \alpha_{23}), \Lambda_{14} = \lambda_{14} + (A + C - p + 1)(\alpha_{12} + \alpha_{23}) \), \( \Lambda_{14, 24} = \lambda_{14, 24} + (A + C - p + 2)(\alpha_{12} + \alpha_{23}), \Lambda_{14, 34} = \lambda_{14, 34} + (A + C - p + 2)(\alpha_{12} + \alpha_{23}) \) and \( \Lambda_{14, 24, 34} = \lambda_{14, 24, 34} + (A + C - p + 1)(\alpha_{12} + \alpha_{23}) \).
2. Characteristic zero case

In the case of characteristic zero, the category of $S$-modules is semi-simple. In particular, $\nabla(\mu) = L(\mu)$ for dominant $S$-weight $\mu$.

The character of the costandard $S$-module $\nabla(\mu)$ is given by $S(\mu)$ and its dimension $\dim \nabla(\mu)$ equals $\frac{1}{2}((l_1 - \mu_2 + 1)(\mu_2 - \mu_3 + 1)(\mu_1 - \mu_3 + 2))$.

2.1. Ground floor. The module $V = H^0_{G_{12}}(\lambda) = L(\lambda)$ is an irreducible $S$-module of dimension $\frac{(C+1)(A+1)(A+C+2)}{2}$.

2.2. First floor. Consider the following vectors in $V \otimes Y$.

- $l_1 = v_{A,0,C,0} \otimes y_{34}$,
- $l_2 = v_{A-1,1,C,0} \otimes y_{34} + v_{A,0,C,0} \otimes y_{24}$ for $A > 0$, and
- $l_3 = v_{A,0,C-1,0} \otimes y_{34} + v_{A,0,C-1,1} \otimes y_{24} + v_{A,0,C,0} \otimes y_{14}$ for $C > 0$.

Lemma 2.1. The $S$-module $V \otimes Y$ is isomorphic to the direct sum $L(l_1) \oplus \delta_{23}L(l_2) \oplus \delta_{12}L(l_3)$.

Proof. The vectors $l_1$, $l_2$, and $l_3$ are primitive vectors of weights $(C, A+1)$, $(C+1, A-1)$, and $(C-1, A)$, respectively. The dimensions of $L(l_1)$, $L(l_2)$, and $L(l_3)$ are $\frac{3(C+1)(A+1)(A+C+2)}{2}$ and $\frac{(C+1)(A+1)(A+C+1)}{2}$, respectively, and they add up to the dimension of $V \otimes Y$, which is $\frac{3(C+1)(A+1)(A+C+2)}{2}$. □

The image of the first floor under the action of superderivations is given in the following Proposition.

Proposition 2.2. Let $\phi_1 : V \otimes Y \rightarrow V \otimes Y$ be a morphism of $S$-modules given by $v \otimes y_{ij} \rightarrow (v)_{ij} D_{ij}$. Then the image $\phi_1(V \otimes Y) \cong \delta_{34}L(l_1) \oplus \delta_{23}L(l_2) \oplus \delta_{12}L(l_3)$.

Proof. We compute $\phi_1(l_1) = \omega_{34}l_1$, $\phi_1(l_2) = \omega_{24}l_2$, $\phi_1(l_3) = \omega_{14}l_3$, and the claim follows. □

2.3. Second floor. The action of the superderivations of $14D$, $24D$, and $34D$ on the space $Y$ is supercommutative, meaning that every $y \in Y$ is annihilated by application of maps $14D_{24}D + 24D_{34}D + 34D_{14}D + 21D_{34}D + 34D_{21}D$.

Therefore on the second floor we can consider a map $\phi_2 : V \otimes (Y \wedge Y) \rightarrow V \otimes (Y \wedge Y)$ given by $v \otimes y_{ij} \wedge y_{kl} \rightarrow (v)_{ij} D_{ij} D_{kl}$.

Consider the following vectors in $V \otimes (Y \wedge Y)$.

- $m_1 = v_{A,0,C,0} \otimes (y_{24} \wedge y_{34})$,
- $m_2 = v_{A,0,C-1,0} \otimes (y_{24} \wedge y_{34}) + v_{A,0,C,0} \otimes (y_{14} \wedge y_{34})$,
- $m_3 = v_{A-1,0,C,0} \otimes (y_{24} \wedge y_{34}) + v_{A-1,1,C,0} \otimes (y_{14} \wedge y_{34}) + v_{A,0,C,0} \otimes (y_{14} \wedge y_{24})$.

If $C > 0$, then

$m_3 = (-v_{A,0,C-1,0} + v_{A-1,1,C-1,1}) \otimes (y_{24} \wedge y_{34}) + v_{A-1,1,C,0} \otimes (y_{14} \wedge y_{34}) + v_{A,0,C,0} \otimes (y_{14} \wedge y_{24})$.

Lemma 2.3. The $S$-module $V \otimes (Y \wedge Y)$ is isomorphic to the direct sum $L(m_1) \oplus \delta_{12}L(m_2) \oplus \delta_{23}L(m_3)$.

Proof. The vectors $m_1$, $m_2$, and $m_3$ are primitive vectors of weights $(C + 1, A)$, $(C - 1, A + 1)$, and $(C, A)$, respectively. The dimensions of $L(m_1)$, $L(m_2)$, and $L(m_3)$ are $\frac{(C+2)(A+1)(A+C+2)}{2}$, $\frac{C(A+2)(A+C+2)}{2}$, and $\frac{(C+1)A(A+C+1)}{2}$, respectively, and they add up to the dimension of $V \otimes (Y \wedge Y)$, which is $\frac{3(C+1)(A+1)(A+C+2)}{2}$. □

The image of the second floor under the action of superderivations is given in the following Proposition.
Proposition 2.4. Let \( \phi_2 : V \otimes (Y \wedge Y) \to V \otimes (Y \wedge Y) \) be a morphism of \( S \)-modules given by \( v \otimes (y_{ij} \wedge y_{kl}) \mapsto (v)_{ij}D_{kl}D \). Then the image \( \phi_2(V \otimes (Y \wedge Y)) = \delta_{24}^3L(m_1) \oplus \delta_{12}^4\delta_{34}^4L(m_2) \oplus \delta_{23}^4\delta_{14}^4\omega_{24}^4L(m_3) \).

Proof. We compute \( \phi_2(m_1) = \omega_{24}\omega_{34}^3m_1, \phi_2(m_2) = \omega_{14}\omega_{34}^3m_2, \phi_2(m_3) = \omega_{14}\omega_{24}^3m_3 \), and the claim follows.

2.4. Third floor. The module \( Y \wedge Y \wedge Y \) is irreducible of highest weight \( y_{14} \wedge y_{24} \wedge y_{34} \) and the module \( V \otimes (Y \wedge Y \wedge Y) \) is irreducible of the highest weight \( l = v_{\lambda,0,C,0} \otimes y_{14} \wedge y_{24} \wedge y_{34} \).

On the third floor we consider a map \( \phi_3 : V \otimes (Y \wedge Y \wedge Y) \to V \otimes (Y \wedge Y \wedge Y) \) given by \( v \otimes (y_{ij} \wedge y_{kl} \wedge y_{mn}) \mapsto (v)_{ij}D_{kl}D_{mn}D \).

Proposition 2.5. The image \( \phi_3(V \otimes (Y \wedge Y \wedge Y)) = \delta_{14}^3\delta_{34}^4V \otimes (Y \wedge Y \wedge Y) \).

Proof. The morphism \( \phi_3 \) is given by \( \phi_3(l) = \omega_{14}\omega_{24}\omega_{34}^3l \).

2.5. Simple module \( L_{S(3|1)}(\lambda) \). Combining previous results we obtain the following theorem.

Theorem 2.6. The simple module \( L_{S(2|2)}(\lambda) \), viewed as an \( S \)-module, is isomorphic to the direct sum \( V \oplus \phi_1(F_1(\lambda)) \oplus \phi_2(F_2(\lambda)) \oplus \phi_3(F_3(\lambda)) \), where the images \( \phi_1(F_1(\lambda)), \phi_2(F_2(\lambda)) \) and \( \phi_3(F_3(\lambda)) \), respectively, are described in Propositions 2.2, 2.4 and 2.5, respectively.

Corollary 2.7. The induced module \( H_G^0(\lambda) \) is isomorphic to \( L_{S(3|1)}(\lambda) \) if and only if \( \lambda \) is typical which happens if and only if \( \lambda_3 > 0 \).

In order to relate the last results to Hook Schur functions, we need to explain how a simple module \( L_{S(3|1)}(\lambda) \) corresponds to a \((3,1)\)-hook partition \( \gamma = (\gamma_1,\ldots,\gamma_k) \).

The correspondence is such that \( \gamma_1 = \lambda_1, \gamma_2 = \lambda_2, \gamma_3 = \lambda_3 \) and the partition \((\gamma_4,\ldots,\gamma_k)\) is the transpose of \((\lambda_4)\).

Let \( \mu = (\mu_1,\mu_2,\mu_3|\mu_4) \) be a dominant \( GL(3|1) \)-weight. Denote by \( S(\mu) = S_{\mu_1,\mu_2,\mu_3}(x_1,x_2,x_3)S_{\mu_4}(y_1) = S_{\mu_1,\mu_2,\mu_3}(x_1,x_2,x_3)y_1^{\mu_4} \), the product of the Schur functions corresponding to \( \mu_+ = (\mu_1,\mu_2,\mu_3) \) and \( \mu_- = \mu_4 \).

The character of the induced module \( H_G^0(\lambda) \) is given by the formula \( \chi(H_G^0(\lambda)) = (1 + \frac{\mu_1}{x_1})(1 + \frac{\mu_2}{x_2})(1 + \frac{\mu_3}{x_3})S_{\lambda_1,\lambda_2,\lambda_3}(x_1,x_2,x_3)S_{\lambda_4}(y_1) \).

Therefore, we have the following equivalence which strengthens Theorem 6.20 of [1] in the case of \((3,1)\)-hook partitions.

Proposition 2.8. For a \((3,1)\)-hook partition \( \lambda \), the following are equivalent:

1) \( \lambda_3 > 0 \)
2) \( HS_\gamma(x_1,x_2,x_3;y_1) = \chi(H_G^0(\lambda)) \)
3) \( H_G^0(\lambda) \) is isomorphic to \( L_{S(3|1)}(\lambda) \).

Proof. If \( \lambda_3 > 0 \), then in the notation of Theorem 6.20 of [1] we have \( \chi(H_G^0(\lambda)) = (x_1+y_1)(x_2+y_1)(x_3+y_1)s_\mu(x_1,x_2,x_3)s_\nu(y_1) \) and \( HS_\gamma(x_1,x_2,x_3;y_1) = \chi(H_G^0(\lambda)) \).

The remaining statements follow from Corollary 2.7.

3. \( S \)-module structure of \( H_G^0(\lambda) \) for restricted \( \lambda \)

Throughout the following three sections assume that \( p > 2 \). In addition to superderivations \( i_{ij}D \), already considered in Section 1, the distribution algebra
$Dist(G)$ also contains generators of type \(\partial_i D^{(h)} = \frac{\partial_i D^h}{h!}\) for \(1 \leq i, j \leq 3\), and
\[(i, D) = \frac{\partial_i D_{(i, D-1)\ldots(i, D-h+1)}}{h!},\]
where \(h = p^k\) for \(k \geq 1\) (See Section 4 of [10]).

In the next two sections we will assume that \(\lambda\) is restricted. Therefore we need not worry about actions of such superderivations because they will act trivially on vectors of restricted weights.

Our goal in this section is to describe the \(S\)-module structure of \(H^0_G(\lambda)\) for a restricted weight \(\lambda\). We will again use formulas from Section 1 extensively. Additionally, all primitive elements exhibited in Section 2 would remain primitive even in the case of positive characteristic, and their images under the morphisms \(\phi_1\), \(\phi_2\) and \(\phi_3\) are given by the formulas from Section 2.

However, the category of \(S\)-modules is not semisimple in the case \(p > 2\); there are additional primitive vectors and the \(S\)-module structure of floors of \(H^0_G(\lambda)\) is more complicated than in the case of characteristic zero.

3.1. Ground floor. We will need the following fact about \(SL(3)\)-modules. For details, see [11]. If \(\lambda\) lies in the fundamental (downward) alcove \(A_0\) or on the walls of the adjacent upward alcove \(A_1\) of the affine Weyl group, then \(\nabla_{SL(3)}(\lambda) = L(\lambda)\). If \(\lambda\) lies in the interior of \(A_1\), then \(\nabla_{SL(3)}(\lambda)\) is filtered by \(L(\lambda)\) and \(L(\lambda')\), where \(\lambda'\) is the reflection of \(\lambda\) along the wall separating \(A_0\) and \(A_1\). Here \(L(\lambda)\) is the socle of \(\nabla_{SL(3)}(\lambda)\) and \(L(\lambda')\) is the top of \(\nabla_{SL(3)}(\lambda)\), and \(\chi(L(\lambda)) = \chi(L(\lambda'))\). In what follows denote \(\chi(L(\lambda))\) by \(SS(\lambda)\). As a consequence, the characters of simple \(SL(3)\)-modules are described as follows.

**Proposition 3.1.** Let \(\lambda = (\lambda_1, \lambda_2, \lambda_3)\) be a dominant \(SL(3)\)-weight and \(\lambda_1 - \lambda_2 = C, \lambda_2 - \lambda_3 = A, \) where \(C, A < p\).

If \(0 < A, C < p - 1\) and \(C + A > p - 2\) (that is \((C, A)\) in the interior of \(A_1\)), then the character \(\chi(L(\lambda))\) of the simple \(SL(3)\)-module \(L(\lambda)\) is given as

\[
\chi(L(\lambda)) = P(C, A, \lambda_3) - P(p - 2 - A, p - 2 - C, C + A - p + 2 + \lambda_3)
\]

and \(\dim L(\lambda) = \frac{(C+1)(A+1)(A+C+2)}{2} - \frac{(p-1-A)(p-1-C)(2p-2-C-A)}{2}\).

Otherwise, that is if \(C = p - 1, A = p - 1\) or \(C + A \leq p - 2\), then

\[
\chi(L(\lambda)) = P(C, A, \lambda_3)
\]

and \(\dim L(\lambda) = \frac{(C+1)(A+1)(A+C+2)}{2}\).

Although we know the \(S\)-module structure of the module \(V = H^0_G(\lambda)\), for further computations we need to find a primitive vector of weight \(\lambda = (\lambda_3 + p - 2, \lambda_2, \lambda_1 - p + 2)\) explicitly.

Assume \(0 < C, A < p - 1\) and \(A + C > p - 2\) and denote \(k = A + C - p + 2 > 0\). Then \(\lambda = k\alpha_{12} + k\alpha_{23}\).

The dimension of the weight space \(V_{\lambda + k\alpha_{12} + (k-1)\alpha_{23}}\) is \(k\) and its basis consists of vectors \(v_{A-b, b, C-k, b+1}\) for \(b = 0, \ldots, k - 1\).

The dimension of the weight space \(V_{\lambda + k\alpha_{12} + \alpha_{23}}\) is \(k + 1\) and its basis consists of vectors \(v_{A-b, b, C-k, b}\) for \(b = 0, \ldots, k\).

Since \((v_{A-b, b, C-k, b+1})_{23}D = (A-b)v_{A-b-1, b+1, C-k, b+1} + (b+1)v_{A-b, b, C-k, b}\), the image of \(V_{\lambda + k\alpha_{12} + (k-1)\alpha_{23}}\) under \(23D\) is \(k\)-dimensional. Since the weight space of \(L(\lambda)_{\lambda + k(\alpha_{12} + \alpha_{23})}\) is also \(k\)-dimensional, we can choose any vector from \(V_{\lambda + k\alpha_{12} + k\alpha_{23}}\).
that is not in $(V_{\lambda+k_{a_{12}}+(k-1)\alpha_{24}})_{25} D$ as a primitive vector in $V$ if weight $\lambda$. Obviously, such a choice is not canonical and we will choose this vector $\overline{v}$ as

$$\overline{v} = v_{A,0,p-2,a,0}.$$ 

**Proposition 3.2.** If $C = p - 1$, $A = p - 1$ or $C + A \leq p - 2$, then the $S$-module $V = L(v)$. If $0 < A, C < p - 1$ and $C + A > p - 2$, then the $S$-composition series for $V$ is

$$V = \frac{L(\overline{v})}{L(v)}.$$

**3.2. First floor.** If $0 < A, C < p - 1$ is such that $C + A > p - 2$, then $V \otimes Y$ contains a submodule $M = \langle v \rangle Y$ and the factormodule $(V \otimes Y)/M \cong L(\overline{v}) \otimes Y$.

If $C > 0$, then denote $I_1 = v_{A,0,c-1,0} \otimes y_{34}$.

If $0 < A, C < p - 1$ is such that $C + A > p - 2$, then define the following elements of $V \otimes Y$.

$$I_1 = v_{A,0,p-2-A,0} \otimes y_{34} + v_{A,0,p-2-A,1} \otimes y_{24} + v_{A,0,p-1-A,0} \otimes y_{14},$$

$$I_2 = (A + 1)v_{A-1,1,p-2-A,0} \otimes y_{34} - (C + 1)v_{A,0,p-2-A,0} \otimes y_{24}$$

$$(A + C + p - 2)v_{A-1,1,p-2-A,1} \otimes y_{24} + (A + C + p - 2)v_{A-1,1,p-1-A,0} \otimes y_{14}$$

for $C < p - 2$,

$$I_3 = (C + A - p - 3)v_{A,0,p-2-A,0} \otimes y_{14} + (C + A - p + 3)v_{A,0,p-3-A,1} \otimes y_{24}$$

$$+ v_{A,0,p-3-A,0} \otimes y_{34}$$


The weights of $I_1$, $I_2$ and $I_3$ are $(p - 2 - A, p - 1 - C)$, $(p - 1 - A, p - 3 - C)$ and $(p - 3 - A, p - 2 - C)$, respectively.

**Lemma 3.3.** Assume $A, C < p - 1$. If $A + C < p - 2$, then the $S$-module $V \otimes Y$ is isomorphic to $N_1 \oplus N_2 \oplus N_3$, where $N_1 = L(I_1)$, $N_2 = L(I_2)$, and $N_3 = \delta_{12} L(I_3)$.

If $A + C = p - 2$, then the $S$-module $V \otimes Y$ is isomorphic to $N_1 \oplus N_2$, where

$$N_1 = L(I_1)$$

if $A = p - 2$ and $N_1 \cong L(l_1)$ if $A \neq p - 2$ and $N_2 = \delta_{23} L(l_2)$.

If $A + C = p - 1$, then the $S$-module $V \otimes Y$ is isomorphic to $N_1 \oplus N_2 \oplus N_3$, where

$$N_1 = L(l_1)$$

if $A = p - 2$ and $N_1 \cong L(l_1)$ if $A < p - 2$; $N_2 = L(l_2)$ if $C = p - 2$

and $N_2 \cong L(I_2)$ if $C < p - 2$; and $N_3 = L(I_1)$.

If $A + C > p - 1$, then the $S$-module $V \otimes Y$ is isomorphic to $N_1 \oplus N_2 \oplus N_3$, where

$$N_1 = L(I_1)$$

if $A = p - 2$ and $N_1 \cong L(l_1)$ if $A < p - 2$; $N_2 = L(l_2)$ if $C = p - 2$

and $N_2 \cong L(I_2)$ if $C < p - 2$; and $N_3 \cong L(I_1)$.
Additionally, \((v) \otimes Y\) is the largest submodule of \(V \otimes Y\) filtered by simple modules \(L(l_1), L(l_2), L(l_3)\) and \(L(l_4)\), and \((V \otimes Y)/(v) \otimes Y\) is the largest factormodule of \(V \otimes Y\) filtered by simple modules \(L(\bar{l}_1), L(\bar{l}_2)\) and \(L(\bar{l}_3)\).

Proof. Since \((l_i)_{21}D = (l_3)_{32}D = 0\) for \(i = 1, 2, 3\), we have that the vectors \(l_1\) and \(l_2\) are maximal primitive vectors. We verify that vectors \(l_1\) and \(l_2\) belong to \((v) \otimes Y\).

The vector \(l_3\) belongs to \((v) \otimes Y\) if and only if \(A + C \neq p - 1\). If \(A + C = p - 1\), then \(l_3 = \bar{l}_1\).

If \(A + C > p - 2\), then \((\bar{l}_i)_{21}D, (\bar{l}_i)_{32}D \in (v) \otimes Y\) for \(i = 1, 2, 3\) which implies that vectors \(\bar{l}_1, \bar{l}_2\) and \(\bar{l}_3\) are primitive vectors. Additionally, either \((\bar{l}_1)_{21}D \neq 0\) or \((\bar{l}_1)_{32}D \neq 0\) for each \(i = 1, 2, 3\) shows that none of \(l_i\) is a maximal primitive vector. If \(A + C \neq p - 2\), then no pair of vectors \(l_1, l_2\) or \(l_3\) are linked and no pair of vectors \(\bar{l}_1, \bar{l}_2, \bar{l}_3\) are linked. This implies that there are only trivial extensions between each pair of simple modules \(L(l_1), L(l_2)\) and \(L(l_3)\) and only trivial extensions between each pair of simple modules \(L(\bar{l}_1), L(\bar{l}_2)\) and \(L(\bar{l}_3)\). Since \(l_1\) is linked with \(l_3\), \(l_2\) is linked with \(l_2\) and \(\bar{l}_3\) is linked with \(\bar{l}_1\), we obtain the indicated extensions:

\[
\begin{array}{ccc}
L(l_3) & L(\bar{l}_2) & L(\bar{l}_1) \\
L(l_2) & & \\
L(l_1) & & \\
\end{array}
\]

The dimension count completes the proof in the case \(A + C \neq p - 2, p - 1\).

The case \(A + C = p - 1\) differs from the case \(A + C > p - 1\) only by the absence of the simple \(L(l_3)\) since \(l_3 \notin (v) \otimes Y\). The dimension count completes the proof in this case as well.

If \(A = p - 2\) and \(C = 0\), the dimension count confirms that \(V \otimes Y \cong L(l_1) \oplus L(l_2)\). If \(A = p - 2\) but \(A < p - 2\), then the vectors \(l_1\) and \(l_3\) are linked. Since \(l_4 \notin \langle l_1 \rangle\) and \((l_4)_{21}D = 0\), \((l_4)_{31}D = (l_1)_1\), we get that \(l_4\) is a primitive vector. Moreover,

\[
(l_1)(23D)_{12}D + (C + 1)_{13}D = -C l_3 \text{ shows that } \langle l_4 \rangle = L(l_1) . \text{ The dimension count concludes the proof in the case } A + C = p - 2. \]

If \(A = p - 1\), denote \(l_5 = v_{p-1,0,0} \otimes y_{24}\).

Lemma 3.4. If \(A = p - 1\) and \(C < p - 1\), then \(V \otimes Y = \langle l_5 \rangle \oplus \delta_{l_2} L(l_3). \) If \(C = p - 2\), then \(\langle l_5 \rangle\) has the \(S\)-composition factors \(L(l_3), L(l_1)\) and \(L(l_2)\). If \(C < p - 2\), then \(\langle l_5 \rangle\) has the \(S\)-composition factors \(L(l_3), L(l_1), L(l_2)\) and \(L(\bar{l}_3)\), where the weight of \(l_3\) is \(x_{24}\).

Proof. If \(A = p - 1\) but \(C \neq p - 1\), then the weights of \(l_1\) and \(l_2\) are linked and \(l_3\) lies in a different block. Moreover, \((l_1)_{23}D = -l_2\) and \((l_5)_{21}D = 0\), \((l_5)_{32}D = l_1\) shows that there is a chain of \(S\)-submodules \(\langle l_2 \rangle \subsetneq \langle l_1 \rangle \subsetneq \langle l_5 \rangle\) and that \(l_5\) is a primitive vector of weight \((C + 1, p - 2)\). If \(C = p - 2\), then by the dimension count \(V \otimes Y = \langle l_5 \rangle \oplus L(l_3). \) If \(C < p - 2\), then \(V \otimes Y\) contains another primitive vector \(l_5\) of weight \((0, p - 3 - C)\). Assume now \(C < p - 2\). We will show that the sum \(I\) of the image of \((V \otimes Y)_{2, p - 4 - C}\) under \(\delta_{l_2} D\) and the image of \((V \otimes Y)_{-1, p - 1 - C}\) under \(23D\) covers the whole weight space \((V \otimes Y)_{0, p - 3 - C}\). This will imply that \(l_5 \in \langle l_5 \rangle\) and \(V \otimes Y = \langle l_5 \rangle \oplus \delta_{l_2} L(l_3). \)
The weight space \((V \otimes Y)_{(2,p-4-C)}\) is \((3C + 2)\)-dimensional and its \(K\)-basis consists of vectors \(v_{p-2-i,i+1,1,i-1} \otimes y_{14}\) for \(i = 1, \ldots, C\), \(v_{p-2-i,i+1,0,i} \otimes y_{24}\) for \(i = 0, \ldots, C\), and \(v_{p-3-i,i+2,0,i} \otimes y_{34} i = 0, \ldots, C\).

The weight space \((V \otimes Y)_{(-1,p-1-C)}\) is \((3C + 2)\)-dimensional and its \(K\)-basis consists of vectors \(v_{p-1-i,i,0,i} \otimes y_{14}\) for \(i = 0, \ldots, C\), \(v_{p-2-i,i,0,i+1} \otimes y_{24}\) for \(i = 0, \ldots, C-1\), and \(v_{p-2-i,i,0,i} \otimes y_{34} i = 0, \ldots, C\).

The weight space \((V \otimes Y)_{(0,p-3-C)}\) is \((3C + 3)\)-dimensional and its \(K\)-basis consists of vectors \(v_{p-2-i,i+1,0,i} \otimes y_{14}\) for \(i = 0, \ldots, C\), \(v_{p-2-i,i,0,i} \otimes y_{24}\) for \(i = 0, \ldots, C\), and \(v_{p-3-i,i+1,0,i} \otimes y_{34} i = 0, \ldots, C\).

Since \((v_{p-3-i,i+2,0,i} \otimes y_{34})_{12D} = (i + 2) v_{p-3-i,i+1,0,i} \otimes y_{34}\) for \(i = 0, \ldots, C\), all vectors \(v_{p-3-i,i+1,0,i} \otimes y_{34} \in I\) for \(i = 0, \ldots, C\).

Since \((v_{p-2-i,i,0,i} \otimes y_{34})_{23D} = (p - 2 - i) v_{p-3-i,i+1,0,i} \otimes y_{34} + iv_{p-2-i,i,0,i-1} \otimes y_{34} - v_{p-2-i,i,0,i} \otimes y_{24}\) for \(i = 0, \ldots, C\) we infer that all \(v_{p-2-i,i,0,i} \otimes y_{24} \in I\) for \(i = 0, \ldots, C\).

Since \((v_{p-2-i,i+1,0,i} \otimes y_{24})_{12D} = (i + 1) v_{p-2-i,i,0,i} \otimes y_{24} - v_{p-2-i,i+1,0,i} \otimes y_{14}\) for \(i = 0, \ldots, C\) we infer that all \(v_{p-2-i,i+1,0,i} \otimes y_{14} \in I\) for \(i = 0, \ldots, C\).

If \(C = p - 1\), denote \(l_6 = v_{A-1,1,p-2,1} \otimes y_{34} + v_{A,0,p-2,1} \otimes y_{24}\).

**Lemma 3.5.** If \(C = p - 1\) and \(A = 0\), then \(V \otimes Y \cong L(l_1) \oplus L(l_3)\). If \(C = p - 1\) and \(0 < A < p - 1\), then \(V \otimes Y = L(l_1) \oplus (l_6)\). The module \(l_6\) has the \(S\)-composition factors \(L(l_6), L(l_2), L(l_3)\) and \(L(l_6)\), where the weight of \(l_6\) is \(\tilde{\lambda}_{14}\).

**Proof.** If \(C = p - 1\) and \(A = 0\), then \(l_1\) and \(l_3\) belong to a different block and a dimension count shows \(V \otimes Y \cong L(l_1) \oplus L(l_3)\). Assume \(C = p - 1\) but \(0 < A < p - 1\). Then the weights of \(l_2\) and \(l_3\) are linked and \(l_1\) lies in a different block.

Moreover, \((l_2)_{12D} = -l_3\) and \((l_6)_{21D} = l_2\). \((l_6)_{32D} = 0\) shows that there is a chain of \(S\)-submodules \((l_3) \subseteq (l_2) \subseteq (l_6)\) and that \(l_6\) is a primitive vector of weight \((p - 2, A)\).

The module \(V \otimes Y\) contains another primitive vector \(l_6\) of weight \((p - 2 - A, 0)\). We will show that the sum \(I\) of the image of \((V \otimes Y)_{(p-A, -1)}\) under \(12D\) and the image of \((V \otimes Y)_{(p-3-A,2)}\) under \(23D\) covers the whole weight space \((V \otimes Y)_{(p-2-A,0)}\).

This will imply that \(l_6 \in (l_6)\) and \(V \otimes Y = L(l_1) \oplus (l_6)\).

The weight space \((V \otimes Y)_{(p-A, -1)}\) is \((3A+1)\)-dimensional and its \(K\)-basis consists of vectors \(v_{A-i,i,p-A,A} \otimes y_{14}\) for \(i = 1, \ldots, A\), \(v_{A-i,i,p-A-1,A} \otimes y_{24}\) for \(i = 0, \ldots, A\), and \(v_{A-i,i,p-1-A,i} \otimes y_{34} i = 1, \ldots, A\).

The weight space \((V \otimes Y)_{(p-3-A,2)}\) is \((3A+1)\)-dimensional and its \(K\)-basis consists of vectors \(v_{A-i,i,p-1-A,i} \otimes y_{14}\) for \(i = 0, \ldots, A-1\), \(v_{A-i,i,p-2-A,i+2} \otimes y_{24}\) for \(i = 0, \ldots, A-1\), and \(v_{A-i,i,p-2-A,i+1} \otimes y_{34} i = 0, \ldots, A\).

The weight space \((V \otimes Y)_{(p-2-A,0)}\) is \((3C + 3)\)-dimensional and its \(K\)-basis consists of vectors \(v_{A-i,i,p-1-A,i} \otimes y_{14}\) for \(i = 0, \ldots, A\), \(v_{A-i,i,p-2-A,i+1} \otimes y_{24}\) for \(i = 0, \ldots, A\), and \(v_{A-i,i,p-2-A,i} \otimes y_{34} i = 0, \ldots, A\).

Since \((v_{A-i,i,p-A,i} \otimes y_{14})_{12D} = (p - A + i) v_{A-i,i,p-A-1,A} \otimes y_{14} - iv_{A-i,i,p-1-A-1,A} \otimes y_{14}\) for \(i = 1, \ldots, A\), all vectors \(v_{A-i,i,p-A,i} \otimes y_{14} \in I\) for \(i = 0, \ldots, A\). Since \((v_{A-i,i,p-1-A,A} \otimes y_{14})_{23D} = v_{0,A,p-1-A,A} \otimes y_{14} + Av_{1,A-1,p-1-A-1,A} \otimes y_{14}\), the vector \(v_{0,A,p-1-A,A} \otimes y_{14} \in I\).

Since \((v_{A-i,i,p-1-A,i} \otimes y_{24})_{12D} = (p - 1 + A) v_{A-i,i,p-2-A,i+1} \otimes y_{24} - iv_{A-i,i,p-1-A,i} \otimes y_{24} - v_{A-i,i,p-2-A,i} \otimes y_{14}\) for \(i = 0, \ldots, A\) we infer that all \(v_{A-i,i,p-2-A,i+1} \otimes y_{24} \in I\) for \(i = 0, \ldots, A\).
Since \((v_{A\cdot i,i,p-2-A,i+1} \otimes y_{34})_{23} D = (A - i)v_{A\cdot i-i,1,p-2-A,i+1} \otimes y_{34} + (i + 1)v_{A\cdot i,i,p-2-A,i} \otimes y_{34} - v_{A\cdot i,i,p-2-A,i+1} \otimes y_{34}\) for \(i = 0, \ldots, A\) we infer that all \(v_{A\cdot i,i,p-2-A,i} \otimes y_{34} \in I\) for \(i = 0, \ldots, A\).

If \(A = C = p - 1\), define
\[
l_7 = -2v_{p-1,0,p-1,0} \otimes y_{34} - v_{p-1,0,p-2,1} \otimes y_{24} + v_{p-2,1,p-2,1} \otimes y_{34} - 2v_{p-1,0,p-2,0} \otimes y_{34},
\]
\[
l_8 = v_{p-1,0,p-1,0} \otimes y_{24},
\]
\[
l_9 = v_{p-1,0,p-1,0} \otimes y_{34}.
\]

**Lemma 3.6.** If \(A = p - 1\) and \(C = p - 1\), then the \(S\)-module
\[
\begin{align*}
L(l_9) & \quad L(l_8) \\
V \otimes Y & \cong L(l_7) \\
L(l_2) & \quad L(l_1) \\
L(l_3) & \quad L(l_1)
\end{align*}
\]
Proof. Since \((l_1)_{23} D = -l_2\) and \((l_2)_{12} D = l_3\), we get \(l_1 \cong L(l_2)\). Since
\[
(l_7)_{23} D = l_2\) and \((l_7)_{32} D = 0,\) the vector \(l_7\) is primitive. Since \((l_8)_{21} D = 0\) and \((l_8)_{32} D = l_1,\) the vector \(l_8\) is primitive. Since \((l_8)_{12} D = l_7 - (l_1)_{13} D,\) we infer that
\[
\begin{align*}
L(l_8) & \quad L(l_7) \\
\langle l_8 \rangle & \cong L(l_2) \\
L(l_3) & \quad L(l_1)
\end{align*}
\]
shows that the vector \(l_9\) is primitive. A dimension count concludes the proof. \(\square\)

3.3. Second floor. If \(\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) is a dominant weight, then \(\tilde{\lambda} = (\lambda_1, \lambda_1 - \lambda_2 + \lambda_3, \lambda_3, \lambda_4)\) is also a dominant weight. The corresponding \(SL(3)\)-weight of \(\tilde{\lambda} = (\lambda_1 - \lambda_2, \tilde{\lambda}_2 - \tilde{\lambda}_3) = (\lambda_2 - \lambda_3, \lambda_1 - \lambda_2),\) which is the transpose of the \(SL(3)\)-weight corresponding to \(\lambda.\)

Define a linear map \(T_0 : H^0_{G_{ev}}(\lambda) \to H^0_{G_{ev}}(\tilde{\lambda})\) by \(T_0(v_{a,b,c,d}) = v_{c,d,a,b}\) and a linear map \(T_1 : Y \to Y \wedge Y\) by \(T_1(y_{34}) = y_{24} \wedge y_{34},\) \(T_1(y_{24}) = y_{14} \wedge y_{34},\) and \(T_1(y_{14}) = y_{14} \wedge y_{24};\) Further, define a linear map \(T : H^0_{G_{ev}}(\lambda) \otimes Y \to H^0_{G_{ev}}(\tilde{\lambda}) \otimes (Y \wedge Y)\) by \(T(v_{a,b,c,d} \otimes y_{ij}) = T_0(v_{a,b,c,d}) \otimes T_1(y_{ij}).\)

**Lemma 3.7.** The map \(T\) induces a bijection on primitive \(S\)-vectors of \(H^0_{G_{ev}}(\lambda) \otimes Y\) and \(H^0_{G_{ev}}(\tilde{\lambda}) \otimes (Y \wedge Y)\) and an isomorphism of their lattices of \(S\)-submodules.

Proof. Using Lemma 1.2 we verify that \(T_0(v_{a,b,c,d}d_{12} D) = T_0(v_{a,b,c,d})_{23} D, T_0(v_{a,b,c,d})_{23} D = T_0(v_{a,b,c,d})_{12} D, T_0(v_{a,b,c,d})_{21} D = -T_0(v_{a,b,c,d})_{13} D, T_0(v_{a,b,c,d})_{32} D = T_0(v_{a,b,c,d})_{21} D, T_0(v_{a,b,c,d})_{31} D = -T_0(v_{a,b,c,d})_{31} D.\)
Therefore $T_0$ induces a bijection on primitive $S$-vectors of $H_{G_{ee}}^0(\lambda)$ and $H_{G_{ee}}^0(\tilde{\lambda})$ and an isomorphism of their lattices of $S$-submodules.

Using Lemma 1.1 we verify that $T_1(y_{ij}D) = T_1(y_{ij}D)$, $T_1(y_{ij}23D) = T_1(y_{ij}12D)$, $T_1(y_{ij}13D) = -T_1(y_{ij}13D)$, $T_1(y_{ij}32D) = T_1(y_{ij}32D)$, and $T_1(y_{ij}31D) = -T_1(y_{ij}31D)$.

Therefore the map $T$ induces a bijection on primitive $S$-vectors of $H_{G_{ee}}^0(\lambda) \otimes Y$ and $H_{G_{ee}}^0(\tilde{\lambda}) \otimes (Y \wedge Y)$ and an isomorphism of their lattices of $S$-submodules. □

Using the last Lemma we can describe the $S$-module structure of the second floor of $H_{G}^0(\lambda)$ using the $S$-module structure of the first floor of $H_{G}^0(\lambda)$.

We start with the following primitive vectors in $H_{G_{ee}}^0(\tilde{\lambda}) \otimes Y$:

- $\tilde{1}_1 = \tilde{v}_{C,0,A,0} \otimes y_{34}$,
- $\tilde{1}_2 = \tilde{v}_{C-1,1,A,0} \otimes y_{34} + \tilde{v}_{C,0,A,0} \otimes y_{24}$ for $C > 0$,
- $\tilde{1}_3 = \tilde{v}_{C,0,A-1,0} \otimes y_{34} + \tilde{v}_{C,0,A-1,1} \otimes y_{24} + \tilde{v}_{C,0,A,0} \otimes y_{14}$ for $A > 0$,

- $\tilde{7}_1 = (C + 1)v_{C-1,1,A,0} \otimes y_{34} - (A + 1)v_{C,0,p-2,0} \otimes y_{24}$ for $A < p - 2$,
- $\tilde{7}_2 = (C + A - p + 3)v_{C,0,p-2,0} \otimes y_{14} + (C + A - p + 3)v_{C,0,p-3,1} \otimes y_{24} + \tilde{v}_{C,0,p-3,0} \otimes y_{34}$ for $C < p - 2$.

The images under $T$ are:

- $m_1 = v_{A,0,C,0} \otimes (y_{24} \wedge y_{34})$,
- $m_2 = v_{A,0,C-1,1} \otimes (y_{24} \wedge y_{34}) + v_{A,0,C,0} \otimes (y_{14} \wedge y_{34})$ for $C > 0$,
- $m_3 = v_{A-1,0,C,0} \otimes (y_{24} \wedge y_{34}) + v_{A-1,1,C,0} \otimes (y_{14} \wedge y_{34}) + v_{A,0,C,0} \otimes (y_{14} \wedge y_{24})$ for $A > 0$,

- $m_4 = v_{A-1,0,C,0} \otimes (y_{24} \wedge y_{34})$ for $A > 0$,
- $m_5 = v_{A,0,p-1,0} \otimes (y_{14} \wedge y_{34})$,
- $m_6 = v_{p-2,1,C-1,1} \otimes (y_{24} \wedge y_{34}) + v_{p-2,1,C,0} \otimes (y_{14} \wedge y_{34})$,
- $m_7 = -2v_{p-1,0,p-1,0} \otimes (y_{14} \wedge y_{24}) - 2v_{p-1,0,p-2,0} \otimes y_{34}$, $m_8 = v_{p-1,0,p-1,0} \otimes y_{14}$, and

Consequently, we obtain the following Lemmas.
Lemma 3.8. Assume \( A, C < p - 1 \). If \( A + C < p - 2 \), then the \( S \)-module \( V \otimes (Y \wedge Y) \) is isomorphic to \( M_1 \oplus M_2 \oplus M_3 \), where \( M_1 = L(m_1), M_2 = \delta_{12}L(m_2), \) and \( M_3 = \delta_{23}L(m_3) \).

If \( A + C = p - 2 \), then the \( S \)-module \( V \otimes (Y \wedge Y) \) is isomorphic to \( M_1 \oplus M_2 \), where

\[
M_1 = L(m_1) \text{ if } C = p - 2 \quad \text{and} \quad M_1 \cong L(m_1) \quad \text{if } C \neq p - 2 \quad \text{and} \quad M_2 = \delta_{12}L(m_2).
\]

If \( A + C = p - 1 \), then the \( S \)-module \( V \otimes (Y \wedge Y) \) is isomorphic to \( M_1 \oplus M_2 \oplus M_3 \), where \( M_1 = L(m_1) \) if \( C = p - 2 \) and \( M_1 \cong L(m_1) \), if \( C < p - 2 \); \( M_2 = L(m_2) \), if \( A < p - 2 \); and \( M_3 = L(m_3) \).

Additionally, \( (v) \otimes (Y \wedge Y) \) is the largest submodule of \( V \otimes (Y \wedge Y) \) filtered by simple modules \( L(m_1), L(m_2), L(m_3) \) and \( L(m_4) \) and \( V \otimes (Y \wedge Y) / ((v) \otimes (Y \wedge Y)) \) is the largest factormodule of \( V \otimes (Y \wedge Y) \) filtered by simple modules \( L(m_1), L(m_2), L(m_3) \).

Lemma 3.9. If \( C = p - 1 \) and \( A < p - 1 \), then \( V \otimes (Y \wedge Y) = \langle m_5 \rangle \oplus \delta_{23}L(m_3) \).

If \( A = p - 2 \), then \( \langle m_5 \rangle \) has the \( S \)-composition factors \( L(m_5), L(m_1) \) and \( L(m_2) \). If \( A < p - 2 \), then \( \langle m_5 \rangle \) has the \( S \)-composition factors \( L(m_5), L(m_1), L(m_2) \) and \( L(m_3) \), where the weight of \( m_5 \) is \( \lambda_{14,34} \).

Lemma 3.10. If \( A = p - 1 \) and \( C = 0 \), then \( V \otimes (Y \wedge Y) \cong L(m_1) \oplus L(m_3) \). If \( A = p - 1 \) and \( 0 < C < p - 1 \), then \( V \otimes (Y \wedge Y) = L(m_1) \oplus \langle m_6 \rangle \). The module \( \langle m_6 \rangle \) has the \( S \)-composition factors \( L(m_6), L(m_2), L(m_3) \) and \( L(m_5) \), where the weight of \( m_6 \) is \( \lambda_{14,24} \).

Lemma 3.11. If \( A = p - 1 \) and \( C = p - 1 \), then the \( S \)-module

\[
\begin{align*}
&V \otimes (Y \wedge Y) \cong L(m_1) & L(m_7) \\
&L(m_9) & & L(m_7) \\
&L(m_8) & & L(m_7) \\
&L(m_2) & & L(m_7) \\
&L(m_3) & & L(m_7)
\end{align*}
\]
3.4. **Third floor.** Since the $S$-module $Y \wedge Y \wedge Y$ is one-dimensional generated by $y_{14} \wedge y_{24} \wedge y_{34}$ of weight $(0, 0)$, we have $V \otimes (Y \wedge Y \wedge Y) \cong V$. Using Proposition 3.2 we obtain the following statement.

**Proposition 3.12.** If $C = p - 1$, $A = p - 1$ or $C + A \leq p - 2$, then the $S$-module $V \otimes (Y \wedge Y \wedge Y) = L(v \otimes (y_{14} \wedge y_{24} \wedge y_{34}))$. If $0 < A, C < p - 1$ and $C + A > p - 2$, then the $S$-composition series for $V \otimes (Y \wedge Y \wedge Y)$ is

\[
V \otimes (Y \wedge Y \wedge Y) = \begin{array}{c}
L(v \otimes (y_{14} \wedge y_{24} \wedge y_{34})) \\
L(v \otimes (y_{14} \wedge y_{24} \wedge y_{34}))
\end{array}
\]

4. **Description of a simple module $L_{S(3|1)}(\lambda)$ for restricted $\lambda$**

If $A + C \leq p - 2$ or $A = p - 1$ or $C = p - 1$, then the $S$-module $M = \langle v \rangle \oplus \phi_1(v \otimes Y) \oplus \phi_2(v \otimes (Y \wedge Y)) \oplus \phi_3(v \otimes (Y \wedge Y \wedge Y))$ is isomorphic to $L_{S(3|1)}(\lambda)$. If $L_{S(3|1)}(\lambda)$

$A + C > p - 2$ and $A, C < p - 1$, then the $S$-module $M$ is isomorphic to $L_{S(3|1)}(\lambda)$.

For the sake of completeness, we will determine the $S$-module structure of the whole module $M$ but since $L_{S(3|1)}(\lambda) \cong \langle v \rangle \oplus \phi_1((v \otimes Y) \oplus \phi_2((v \otimes (Y \wedge Y)) \oplus \phi_3((v \otimes (Y \wedge Y \wedge Y))$, it would be enough to restrict to only the corresponding images induced from $\langle v \rangle$.

4.1. **First floor.** Denote $M_v = \langle v \rangle \otimes Y$.

In Proposition 2.2 we have already computed $\phi_1(l_1) = \omega_{14}l_1, \phi_1(l_2) = \omega_{24}l_2$ and $\phi_1(l_3) = \omega_{14}l_3$.

**Lemma 4.1.** If $A, C < p - 1$ and $A + C < p - 2$, then $\phi_1(v \otimes Y) \cong \delta_{34}L(l_1) \oplus \delta_{23}\delta_{24}L(l_2) \oplus \delta_{12}\delta_{14}L(l_3)$.

If $A = p - 2$ and $C = 0$, then $\phi_1(v \otimes Y) \cong \delta_{34}L(l_1) \oplus \delta_{24}L(l_2)$. If $A + C = p - 2,$

\[
A < p - 2 \text{ and } \lambda \text{ is typical or } 24\text{-atypical, then } \phi_1(v \otimes Y) \cong L(l_1) \oplus \delta_{23}\delta_{24}L(l_2).
\]

If $A + C = p - 2$, $A < p - 2$ and $\lambda$ is $(14, 34)$-atypical, then $\phi_1(v \otimes Y) \cong L(l_3) \oplus \delta_{23}L(l_2)$.

If $A, C < p - 1$ and $A + C > p - 2$, then $\phi_1(M_v) \cong \delta_{34}L(l_1) \oplus \delta_{23}\delta_{24}L(l_2) \oplus (1 - \delta_{A+C,p-1})\delta_{12}\delta_{14}L(l_3)$ and $\phi_1((v \otimes Y)/M_v) \cong \delta_{14}L(l_1) \oplus (1 - \delta_{C,p-2})\delta_{24}L(l_2) \oplus (1 - \delta_{A,p-2})\delta_{34}L(l_3)$.

**Proof.** The case $A + C < p - 2$ is straightforward by Lemma 3.3.

In the case $A + C = p - 2$ we have $\omega_{14} = \omega_{34}$ and $\phi_1(l_4) = \omega_{34}l_4 + l_3$. Lemma 3.3 implies the second part of the statement.

Assume $A + C > p - 2$. Then $\phi_1(l_1) = \omega_{14}l_1, \phi_1(l_2) = -\omega_{24}l_2$, and $\phi_1(l_3) = \omega_{34}l_3$. This together with Lemma 3.3 implies the last part of the lemma.

**Lemma 4.2.** Assume $A = p - 1$ and $C < p - 1$. If $\lambda$ is typical or $(14)$-atypical, then $\phi_1(v \otimes Y) \cong \langle l_5 \rangle \oplus \delta_{12}\delta_{14}L(l_3)$. If $\lambda$ is $(24, 34)$-atypical, then $\phi_1(v \otimes Y) \cong L(l_2) \oplus \delta_{12}L(l_3)$.

**Proof.** The statement follows from $\phi_1(l_5) = \omega_{24}l_5 - l_2$ and Lemma 3.4.
Lemma 4.3. If $C = p - 1$ and $A = 0$, then $\phi_1(Y \otimes Y) \equiv \delta_{34}L(l_1) \oplus \delta_{12}\delta_{14}L(l_3)$. If $C = p - 1$, $0 < A < p - 1$ and $\lambda$ is typical or $(34)$-atypical, then $\phi_1(Y \otimes Y) = \delta_{34}L(l_1) \oplus \{l_6\}$. If $C = p - 1$, $0 < A < p - 1$ and $\lambda$ is $(14, 24)$-atypical, then $\phi_1(Y \otimes Y) = L(l_1) \oplus L(l_3)$.

Proof. The statement follows from $\phi_1(l_6) = \omega_{24}l_6 + l_3$ and Lemma 3.5.

Lemma 4.4. If $A = p - 1$, $C = p - 1$ and $\lambda$ is typical, then $\phi_1(Y \otimes Y) \cong V \otimes Y$. If $A = p - 1$, $C = p - 1$ and $\lambda$ is $(14, 24, 34)$-atypical, then $\phi_1(Y \otimes Y) \cong (l_7)$.

Proof. The statement follows from $\phi_1(l_6) = \omega_{14}l_6 + l_7$ and Lemma 3.6.

4.2. Second floor. Denote $N_6 = \langle v \rangle \otimes Y$.

In Proposition 2.4 we have already computed $\phi_2(m_1) = \omega_{24}\omega_{34}m_1$, $\phi_2(m_2) = \omega_{14}\omega_{34}m_2$, $\phi_2(m_3) = \omega_{24}\omega_{24}m_3$.

Lemma 4.5. If $A, C < p - 1$ and $A + C < p - 2$, then $\phi_2(Y \otimes (Y \wedge Y)) \cong \delta_{24}\delta_{34}L(m_1) \oplus \delta_{12}\delta_{14}\delta_{24}L(m_3)$.

If $C = p - 2$ and $A = 0$, then $\phi_2(Y \otimes Y) \cong \delta_{24}\delta_{34}L(m_1) \oplus \delta_{14}\delta_{34}L(m_2)$. If $A + C = p - 2$, $C < p - 2$ and $\lambda$ is typical or $24$-atypical, then $\phi_2(Y \otimes Y) \cong L(m_4)$.

$$\delta_{24}L(m_1) \oplus \delta_{12}L(m_2).$$

If $A + C = p - 2$, $C < p - 2$ and $\lambda$ is $(14, 34)$-atypical, then $L(m_3)$.

Proof. The first part is straightforward by Lemma 3.8.

In the case $A + C = p - 2$ we have $\omega_{14} = \omega_{34}$ and $\phi_2(m_4) = \omega_{24}\omega_{34}m_4 + \omega_{24}m_3$. Lemma 3.3 implies the second part of the statement.

Assume $A, C < p - 1$ and $A + C > p - 2$. Then $\phi_1(m_1) = \omega_{14}\omega_{24}m_1$, $\phi_1(m_2) = -\omega_{14}\omega_{24}m_2$, and $\phi_1(m_3) = \omega_{24}\omega_{34}m_3$ together with Lemma 3.8 implies the second part.

Lemma 4.6. Assume $C = p - 1$ and $A < p - 1$. If $\lambda$ is typical, then $\phi_2(Y \otimes (Y \wedge Y)) \cong \langle m_5 \rangle \oplus \delta_{23}L(m_3)$. If $\lambda$ is atypical, then $\phi_2(Y \otimes (Y \wedge Y)) \cong \delta_{34}L(m_2) \oplus \delta_{23}\delta_{14}\delta_{24}L(m_3)$.

Proof. The statement follows from $\phi_2(m_5) = \omega_{34}\omega_{14}m_5 - \omega_{34}m_5$, $\omega_{14} = \omega_{24}$ and Lemma 3.9.

Lemma 4.7. If $A = p - 1$ and $C = 0$, then $\phi_2(Y \otimes (Y \wedge Y)) \cong \delta_{24}\delta_{34}L(m_1) \oplus \delta_{14}\delta_{24}L(m_3)$. If $A = p - 1$, $0 < C < p - 1$ and $\lambda$ is typical, then $\phi_2(Y \otimes (Y \wedge Y)) = L(m_1) \oplus \langle m_6 \rangle$. If $A = p - 1$, $0 < C < p - 1$ and $\lambda$ is atypical, then $\phi_2(Y \otimes (Y \wedge Y)) \cong \delta_{24}\delta_{34}L(m_1) \oplus \delta_{14}L(m_3)$.

Proof. The statement follows from $\phi_2(m_6) = \omega_{14}\omega_{34}m_6 + \omega_{14}m_3$, $\omega_{24} = \omega_{34}$ and Lemma 3.10.

Lemma 4.8. If $A = p - 1$, $C = p - 1$ and $\lambda$ is typical, then $\phi_2(Y \otimes (Y \wedge Y)) \cong V \otimes (Y \wedge Y)$. If $A = p - 1$, $C = p - 1$ and $\lambda$ is $(14, 24, 34)$-atypical, then $\phi_2(Y \otimes (Y \wedge Y)) \cong L(m_3)$. 

Proof. If λ is typical, then \( \phi_2(m_3) \neq 0 \) and Lemma 3.11 implies that \( \phi_2 \) is injective. If λ is atypical, then the statement follows from \( \phi_2(m_9) = m_3 \) and Lemma 3.11. \( \square \)

4.3. Third floor.

**Lemma 4.9.** If λ is typical, then \( \phi_3(V \otimes (Y \wedge Y \wedge Y)) \cong V \otimes (Y \wedge Y \wedge Y) \). If λ is atypical, then \( \phi_3(V \otimes (Y \wedge Y \wedge Y)) \cong 0 \).

**Proof.** The statement follows from Proposition 3.12, \( \phi_3(l) = \omega_{14} \omega_{24} \omega_{34} l \) and

\[
\phi_3(v_{A, 0, p - 2 - A, 0} \otimes (y_{14} \wedge y_{24} \wedge y_{34})) = \omega_{14} \omega_{24} \omega_{34} v_{A, 0, p - 2 - A, 0} \otimes (y_{14} \wedge y_{24} \wedge y_{34}).
\]

\( \square \)

**Proposition 4.10.** If \( A, C < p - 1 \) and \( A + C \neq p - 2 \), then the simple module \( L_{S(3l)}(\lambda) \) as an \( S \)-module is isomorphic to \( L(v) \oplus \delta_{34} L(l_1) \oplus \delta_{24} \delta_{24} L(l_2) \oplus (1 - \delta_{A + C, p - 1}) \delta_{12} \delta_{14} \delta_{34} L(m_1) \oplus \delta_{12} \delta_{14} \delta_{34} L(m_2) \oplus (1 - \delta_{A + C, p - 1}) \delta_{23} \delta_{14} \delta_{24} L(m_3) \oplus \delta_{14} \delta_{24} \delta_{34} L(l). \)

Assume \( A = 0 \) and \( C = p - 2 \). If λ is typical or 24-atypical, then \( L_{S(3l)}(\lambda) \cong L(v) \oplus \delta_3 L(l_1) \oplus \delta_2 \delta_4 L(l_2) \oplus \delta_4 (m_4) \oplus \delta_2 L(l). \) If λ is (14, 34)-atypical, then \( L_{S(3l)}(\lambda) \cong L(v) \oplus L(l_3). \)

Assume \( A = p - 2 \) and \( C = 0 \). If λ is typical or 24-atypical, then \( L_{S(3l)}(\lambda) \cong L(v) \oplus \delta_3 L(l_1) \oplus \delta_2 \delta_4 L(l_2) \oplus \delta_4 (m_4) \oplus \delta_2 L(l). \) If λ is (14, 34)-atypical, then \( L_{S(3l)}(\lambda) \cong L(v) \oplus L(l_3) \oplus L(l_2) \oplus L(m_3). \)

Assume \( 0 < A, C \) and \( A + C = p - 2 \). If λ is typical or 24-atypical, then \( L_{S(3l)}(\lambda) \cong L(v) \oplus \delta_3 L(l_1) \oplus \delta_2 \delta_4 L(l_2) \oplus \delta_4 (m_4) \oplus L(m_2) \oplus \delta_2 L(l). \) If λ is (14, 34)-atypical, then \( L_{S(3l)}(\lambda) \cong L(v) \oplus L(l_3) \oplus L(l_2) \oplus L(m_3). \)

Assume \( A = p - 1 \) and \( C = 0 \). If λ is typical or 34-atypical, then \( L_{S(3l)}(\lambda) \cong L(v) \oplus \delta_3 L(l_1) \oplus \delta_5 (m_5) \oplus L(m_3) \oplus L(l). \) If λ is (14, 24)-atypical, then \( L_{S(3l)}(\lambda) \cong L(v) \oplus \delta_3 L(l_1) \oplus L(l_3) \oplus L(m_2). \)

Assume \( A = 0 \) and \( C = p - 1 \). If λ is typical, then \( L_{S(3l)}(\lambda) \cong L(v) \oplus \delta_5 (m_5) \oplus L(l_3) \oplus L(l_1) \).

Assume \( A = p - 1 \) and \( C = p - 1 \). If λ is typical, then \( L_{S(3l)}(\lambda) \cong L(v) \oplus \delta_5 (m_5) \oplus \delta_5 (m_5) \oplus L(l_1) \).

If λ is (14, 24, 34)-atypical, then \( L_{S(3l)}(\lambda) \cong L(v) \oplus L(l_3) \oplus L(m_2). \)

**Proof.** The proof follows from Lemmas 3.2, 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9. \( \square \)

5. The Character of the Simple Module \( L_{S(3l)}(\lambda) \)

For the purpose of finding the character of the simple module \( L_{S(3l)}(\lambda) \) and the decomposition numbers for \( S(3l) \) it will be helpful to express explicitly all \( S \)-composition factors (and their multiplicities) of the first and second floors of \( L_{S(3l)}(\lambda) \). We will break it into 15 types.
**Proposition 5.1.** Assume $\lambda$ is restricted. The ground floor $L_0(\lambda)$ and the third floor $L_3(\lambda)$ of the simple module $L_{S(3|1)}(\lambda)$ are given as $L_0(\lambda) = (v)$ and $L_3(\lambda) \cong \delta_{14} \delta_{24} \delta_{34} L(l)$.

The $S$-composition factors of the first floor $L_1(\lambda)$ and the second floor $L_2(\lambda)$ of the simple module $L_{S(3|1)}(\lambda)$ are given as follows.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$L_1(\lambda)$</th>
<th>$L_2(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) $0 &lt; A, C &lt; p - 1$; $A + C \neq p - 1, p - 2$</td>
<td>$L(\lambda_{34}), L(\lambda_{24}), L(\lambda_{14})$</td>
<td>$L(\lambda_{24}, 34), L(\lambda_{14}, 34), L(\lambda_{14, 24})$</td>
</tr>
<tr>
<td>typical</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14 − atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{24})$</td>
<td>$L(\lambda_{24}, 34)$</td>
</tr>
<tr>
<td>24 − atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{14})$</td>
<td>$L(\lambda_{14}, 34)$</td>
</tr>
<tr>
<td>34 − atypical</td>
<td>$L(\lambda_{24}), L(\lambda_{14})$</td>
<td>$L(\lambda_{14, 24})$</td>
</tr>
<tr>
<td>2) $A = 0, 0 &lt; C &lt; p - 2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>typical</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14 − atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{14})$</td>
<td>$L(\lambda_{24}, 34), L(\lambda_{14, 34})$</td>
</tr>
<tr>
<td>24 − atypical</td>
<td>$L(\lambda_{34})$</td>
<td>$L(\lambda_{24, 34})$</td>
</tr>
<tr>
<td>34 − atypical</td>
<td>$L(\lambda_{14})$</td>
<td>$L(\lambda_{14, 34})$</td>
</tr>
<tr>
<td>3) $0 &lt; A &lt; p - 2, C = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>typical</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14 − atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{24})$</td>
<td>$L(\lambda_{24, 34}), L(\lambda_{14, 24})$</td>
</tr>
<tr>
<td>24 − atypical</td>
<td>$L(\lambda_{34})$</td>
<td>$L(\lambda_{24, 34})$</td>
</tr>
<tr>
<td>34 − atypical</td>
<td>$L(\lambda_{14})$</td>
<td>$L(\lambda_{14, 24})$</td>
</tr>
<tr>
<td>4) $A = 0, C = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>typical</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14 − atypical</td>
<td>$L(\lambda_{34})$</td>
<td>$L(\lambda_{24, 34})$</td>
</tr>
<tr>
<td>24 − atypical</td>
<td>$L(\lambda_{34})$</td>
<td>$L(\lambda_{24, 34})$</td>
</tr>
<tr>
<td>34 − atypical</td>
<td>$L(\lambda_{14})$</td>
<td>$L(\lambda_{14, 24})$</td>
</tr>
<tr>
<td>5) $0 &lt; A, C; A + C = p - 2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>typical</td>
<td>$L(\lambda_{34}), L(\lambda_{24}), L(\lambda_{14})^2$</td>
<td>$L(\lambda_{24, 34}), L(\lambda_{14, 34}), L(\lambda_{14, 24})^2$</td>
</tr>
<tr>
<td>(14, 34) − atypical</td>
<td>$L(\lambda_{24}), L(\lambda_{14})$</td>
<td>$L(\lambda_{14, 24})$</td>
</tr>
<tr>
<td>24 − atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{14})^2$</td>
<td>$L(\lambda_{14, 34})$</td>
</tr>
<tr>
<td>6) $A = 0, C = p - 2$</td>
<td></td>
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</tr>
<tr>
<td>typical</td>
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<td></td>
</tr>
<tr>
<td>(14, 34) − atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{14})^2$</td>
<td>$L(\lambda_{24, 34}), L(\lambda_{14, 34})$</td>
</tr>
<tr>
<td>24 − atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{14})^2$</td>
<td>$L(\lambda_{14, 34})$</td>
</tr>
<tr>
<td>7) $A = p - 2, C = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>(14, 34) − atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{24})$</td>
<td>$L(\lambda_{24, 34}), L(\lambda_{14, 24})^2$</td>
</tr>
<tr>
<td>24 − atypical</td>
<td>$L(\lambda_{24})$</td>
<td>$L(\lambda_{14, 24})$</td>
</tr>
<tr>
<td>8) $0 &lt; A, C; A + C = p - 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>typical</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14 − atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{24})$</td>
<td>$L(\lambda_{24, 34}), L(\lambda_{14, 34})$</td>
</tr>
<tr>
<td>24 − atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{24})$</td>
<td>$L(\lambda_{14, 34})$</td>
</tr>
<tr>
<td>34 − atypical</td>
<td>$L(\lambda_{24})$</td>
<td>$L(\lambda_{14, 34})$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$L_1(\lambda)$</td>
<td>$L_2(\lambda)$</td>
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<tr>
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</tr>
<tr>
<td>9) $A = 0, C = p - 1$</td>
<td>typical</td>
<td>$L(\lambda_{34}), L(\lambda_{14})$</td>
</tr>
<tr>
<td></td>
<td>(14, 24) - atypical</td>
<td>$L(\lambda_{31})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$L(\lambda_{14})$</td>
</tr>
<tr>
<td>10) $A = p - 1, C = 0$</td>
<td>typical</td>
<td>$L(\lambda_{34}), L(\lambda_{24}), L(\lambda_{24})^2$</td>
</tr>
<tr>
<td></td>
<td>14 - atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{24})^2, L(\lambda_{24})^3$</td>
</tr>
<tr>
<td></td>
<td>(24, 34) - atypical</td>
<td>$L(\lambda_{24})$</td>
</tr>
<tr>
<td>11) $A = p - 1, C = p - 1$</td>
<td>typical</td>
<td>$L(\lambda_{34}), L(\lambda_{24}), L(\lambda_{24})^2, L(\lambda_{24})^3$</td>
</tr>
<tr>
<td></td>
<td>(14, 24, 34) - atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{24})^2, L(\lambda_{14})^3$</td>
</tr>
<tr>
<td>12) $A = p - 1, C = p - 2$</td>
<td>typical</td>
<td>$L(\lambda_{34}), L(\lambda_{24}), L(\lambda_{24})^2, L(\lambda_{14})$</td>
</tr>
<tr>
<td></td>
<td>(24, 34) - atypical</td>
<td>$L(\lambda_{24})$, $L(\lambda_{14})$</td>
</tr>
<tr>
<td>13) $A = p - 1, 0 &lt; C &lt; p - 2$</td>
<td>typical</td>
<td>$L(\lambda_{34}), L(\lambda_{24}), L(\lambda_{14}), L(\lambda_{14})$</td>
</tr>
<tr>
<td></td>
<td>(14, 24) - atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{24}), L(\lambda_{24})^2, L(\lambda_{14})$</td>
</tr>
<tr>
<td></td>
<td>(24, 34) - atypical</td>
<td>$L(\lambda_{24})$, $L(\lambda_{14})$</td>
</tr>
<tr>
<td>14) $0 &lt; A &lt; p - 2; C = p - 1$</td>
<td>typical</td>
<td>$L(\lambda_{34}), L(\lambda_{24}), L(\lambda_{14}), L(\lambda_{14})$</td>
</tr>
<tr>
<td></td>
<td>(14, 24) - atypical</td>
<td>$L(\lambda_{34}), L(\lambda_{24}), L(\lambda_{14}), L(\lambda_{14})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$L(\lambda_{24})$, $L(\lambda_{14})$</td>
</tr>
</tbody>
</table>

Proof. The proof follows from Lemmas 3.2, 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9.

**Theorem 5.2.** The character $\chi_{S(3|1)}(\lambda)$ of the simple module $L_{S(3|1)}(\lambda)$ for restricted $\lambda$ is given by the following formulas.

Assume $\lambda$ is typical, 14-atypical, 24-atypical or 34-atypical. If $A + C \leq p - 2$ or $A = p - 1$ or $C = p - 1$, then

$$
\chi_{S(3|1)}(\lambda) = S(\lambda) + \delta_{34}S(\lambda_{34}) + \delta_{23}\delta_{24}S(\lambda_{24}) + \delta_{12}\delta_{14}S(\lambda_{14}) + \delta_{24}\delta_{34}S(\lambda_{24,34}) + \delta_{12}\delta_{14}\delta_{34}S(\lambda_{14,34}) + \delta_{23}\delta_{14}\delta_{24}S(\lambda_{14,24}) + \delta_{14}\delta_{24}\delta_{34}S(\lambda_{14,24}).
$$

Moreover, if $A + C > p - 2$ and $A, C < p - 1$, then

$$
\chi_{S(3|1)}(\lambda) = SS(\lambda) + \delta_{34}SS(\lambda_{34}) + \delta_{23}\delta_{24}SS(\lambda_{24}) + \epsilon\delta_{12}\delta_{14}SS(\lambda_{14}) + \delta_{24}\delta_{34}SS(\lambda_{24,34}) + \delta_{12}\delta_{14}\delta_{34}SS(\lambda_{14,34}) + \epsilon\delta_{23}\delta_{14}\delta_{24}SS(\lambda_{14,24}) + \epsilon\delta_{14}\delta_{24}\delta_{34}SS(\lambda_{14,24}),
$$

where $\epsilon = 0$ if $A + C = p - 1$ and $\epsilon = 1$ otherwise.

If $\lambda$ is (14, 34)-atypical or (14, 24, 34)-atypical, then $\chi_{S(3|1)}(\lambda) = S(\lambda) + \delta_{23}S(\lambda_{24}) + \delta_{12}S(\lambda_{14}) + \delta_{23}S(\lambda_{14,24}).$

If $\lambda$ is (14, 24)-atypical, then $\chi_{S(3|1)}(\lambda) = S(\lambda) + S(\lambda_{34}) + \delta_{23}S(\lambda_{14}) + SS(\lambda_{14,34}).$

If $\lambda$ is (24, 34)-atypical, then $\chi_{S(3|1)}(\lambda) = S(\lambda) + SS(\lambda_{24}) + \delta_{12}S(\lambda_{14}) + \delta_{12}S(\lambda_{14,24}).$
Proof: The proof follows from Proposition 5.1 and formulas \( \chi((l_0)) = S(\lambda_{34}) + S(\lambda_{24}), \chi((m_0)) = S(\lambda_{24,34}) + S(\lambda_{14,34}) \).

Finally, using the Steinberg Tensor Product Theorem, for an arbitrary dominant weight \( \lambda \), and for each weight appearing in the composition series of \( L_{S(3|1)}(\lambda) \), \( \lambda = \lambda_r + p \lambda_s \). We will state its typicality and its type.

From the above theorem it is easy to determine the dimension of the simple module \( L_{S(3|1)}(\lambda) \) for restricted \( \lambda \) using the fact that for a weight \( \mu \) with \( C(\mu) = \mu_1 - \mu_2, A(\mu) = \mu_2 - \mu_3 \) we have \( \dim(L(\lambda)) = (A(\mu) + 1)(C(\mu) + 1)(A(\mu) + C(\mu) + 2) \). If \( A(\mu) + C(\mu) \leq p - 2 \) or \( A(\mu) = p - 1 \) or \( C(\mu) = p - 1 \) and \( \dim(L(\lambda)) = \frac{(A(\mu) + 1)(C(\mu) + 1)(A(\mu) + C(\mu) + 2)}{2} - \frac{(p - 1 - A(\mu))(p - 1 - C(\mu))(2p - 2 - A(\mu) - C(\mu))}{2} \). We will leave it to the reader to state formulas for the dimension of \( L_{S(3|1)}(\lambda) \) in each case. Finally, using the Steinberg Tensor Product Theorem, for an arbitrary dominant weight \( \lambda \), the simple module \( L_{S(3|1)}(\lambda) \cong L_{S(3|1)}(\lambda_r) \otimes L(\lambda_s)^F \), where \( \lambda = \lambda_r + p \lambda_s \). We will describe \( \lambda_s \) is restricted and \( F \) denotes the Frobenius shift. (See [9]). Therefore the character and the dimension of an arbitrary simple module \( L_{S(3|1)}(\lambda) \) is determined this way.

6. Decomposition numbers for \( S(3|1) \)

In this section we assume that \( \lambda \) is restricted and \( \lambda_3 > 0 \). We will describe composition factors (the decomposition numbers for \( S(3|1) \)) of costandard modules \( \nabla(\lambda) \) for such hook weight \( \lambda \).

Denote the module \( \langle v \rangle \oplus \langle (v) \otimes Y \rangle \oplus \langle (v) \otimes (Y \wedge Y) \rangle \oplus \langle (v) \otimes (Y \wedge Y \wedge Y) \rangle \) by \( N(\lambda) \). If \( \lambda \) is typical, then \( N(\lambda) \cong L_{S(3|1)}(\lambda) \).

In the following 15 lemmas we determine the composition series of the module \( N(\lambda) \) for atypical \( \lambda \). The basic tool in their proof, which we will use without an explicit reference, is Proposition 5.1. We will use the labeling of types as in Proposition 5.1 and for each weight appearing in the composition series of \( N(\lambda) \) we will state its typicality and its type.

**Lemma 6.1. Assume \( \lambda \) is of type 1.** If \( \lambda \) is 14-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda) \) and \( L_{S(3|1)}(\lambda_{14}) \). If \( \lambda \) is 12-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda) \) and \( L_{S(3|1)}(\lambda_{24}) \). If \( \lambda \) is 34-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda) \) and \( L_{S(3|1)}(\lambda_{34}) \).

**Proof.** Assume \( \lambda \) is 14-atypical. Then \( \lambda_{14} \) is 14-atypical. If \( C > 1 \) and \( A + C \neq p \), then \( L_{S(3|1)}(\lambda_{14}) \) is of type 1. If \( C > 1 \) and \( A + C = p \), then \( L_{S(3|1)}(\lambda_{14}) \) is of type 8. If \( C = 1 \), then \( L_{S(3|1)}(\lambda_{14}) \) is of type 3.

Assume \( \lambda \) is 24-atypical. If \( C \neq p - 2 \), then \( \lambda_{24} \) is 24-atypical; and if \( C = p - 2 \), then \( \lambda_{24} \) is (14, 24)-atypical. If \( A > 1 \) and \( C \neq p - 2 \), then \( \lambda_{24} \) is of type 1). If \( A > 1 \) and \( C = p - 2 \), then \( \lambda_{24} \) is of type 14. If \( A = 1 \), then \( \lambda_{24} \) is of type 2).

Assume \( \lambda \) is 34-atypical. If \( A = p - 2 \) and \( A + C \neq p - 3 \), then \( \lambda_{34} \) is 34-atypical; if \( A + C = p - 3 \), then \( \lambda_{34} \) is (14, 34)-atypical; and if \( A = p - 2 \), then \( \lambda_{34} \) is (24, 34)-atypical. If \( A = p - 2 \) and \( A + C \neq p - 3 \), then \( \lambda_{34} \) is of type 1). If \( A \neq p - 2 \) and \( A + C = p - 3 \), then \( \lambda_{34} \) is of type 5). If \( A = p - 2 \) and \( C \neq p - 2 \), then \( \lambda_{34} \) is of type 13). If \( A = p - 2 \) and \( C = p - 2 \), then \( \lambda_{34} \) is of type 12).}

**Lemma 6.2. Assume \( \lambda \) is of type 2.** If \( \lambda \) is 14-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda) \) and \( L_{S(3|1)}(\lambda_{14}) \). If \( \lambda \) is 24-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda) \) and \( L_{S(3|1)}(\lambda_{24,34}) \). If \( \lambda \) is 34-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda) \) and \( L_{S(3|1)}(\lambda_{34}) \).
Lemma 6.3. Assume λ is of type 3). If λ is 14-atypical, then \(N(\lambda)\) is filtered by \(L_{S(3|1)}(\lambda)\) and \(L_{S(3|1)}(\lambda_{14,24})\). If λ is 24-atypical, then \(N(\lambda)\) is filtered by \(L_{S(3|1)}(\lambda)\) and \(L_{S(3|1)}(\lambda_{24,34})\). If λ is 34-atypical, then \(N(\lambda)\) is filtered by \(L_{S(3|1)}(\lambda)\) and \(L_{S(3|1)}(\lambda_{34})\).

Proof. Assume λ is 14-atypical. Then \(\lambda_{14,24}\) is 24-atypical. If \(A > 1\), then \(L_{S(3|1)}(\lambda_{14,24})\) is of type 3). If \(A = 1\), then \(L_{S(3|1)}(\lambda_{14,24})\) is of type 4).

Assume λ is 24-atypical. Then \(\lambda_{24}\) is 24-atypical. If \(A > 1\), then \(L_{S(3|1)}(\lambda_{24})\) is of type 1). If \(A = 1\), then \(L_{S(3|1)}(\lambda_{24})\) is of type 2) for \(p \neq 3\) and of type 6) for \(p = 3\).

Assume λ is 34-atypical. If \(A \neq p - 3\), then \(\lambda_{34}\) is 34-atypical and \(L_{S(3|1)}(\lambda_{34})\) is of type 3). If \(A = p - 3\), then \(\lambda_{34}\) is (14,34)-atypical and \(L_{S(3|1)}(\lambda_{34})\) is of type 5) for \(p \neq 3\) and of type 7) for \(p = 3\).

Lemma 6.4. Assume λ is of type 4). If λ is 14-atypical, then \(N(\lambda)\) is filtered by \(L_{S(3|1)}(\lambda)\) and \(L_{S(3|1)}(\lambda_{14,24,34})\). If λ is 24-atypical, then \(N(\lambda)\) is filtered by \(L_{S(3|1)}(\lambda)\) and \(L_{S(3|1)}(\lambda_{24,34})\). If λ is 34-atypical, then \(N(\lambda)\) is filtered by \(L_{S(3|1)}(\lambda)\) and \(L_{S(3|1)}(\lambda_{34})\).

Proof. Assume λ is 14-atypical. Then \(\lambda_{14,24,34}\) is 34-atypical and \(L_{S(3|1)}(\lambda_{14,24,34})\) is of type 4).

Assume λ is 24-atypical. If \(p \neq 3\), then \(\lambda_{24,34}\) is 34-atypical and \(L_{S(3|1)}(\lambda_{24,34})\) is of type 2). If \(p = 3\), then \(\lambda_{24,34}\) is (14,34)-atypical and \(L_{S(3|1)}(\lambda_{24,34})\) is of type 6).

Assume λ is 34-atypical. If \(p \neq 3\), then \(\lambda_{34}\) is 34-atypical and \(L_{S(3|1)}(\lambda_{34})\) is of type 3). If \(p = 3\), then \(\lambda_{34}\) is (14,34)-atypical and \(L_{S(3|1)}(\lambda_{34})\) is of type 7).

Lemma 6.5. Assume λ is of type 5). If λ is (14,34)-atypical, then \(N(\lambda)\) is filtered by \(L_{S(3|1)}(\lambda), L_{S(3|1)}(\lambda_{34})\) and \(L_{S(3|1)}(\lambda_{14})\). If λ is 24-atypical, then \(N(\lambda)\) is filtered by \(L_{S(3|1)}(\lambda)\) and \(L_{S(3|1)}(\lambda_{24})\).

Proof. Assume λ is (14,34)-atypical. Then \(\lambda_{34}\) is 34-atypical and \(L_{S(3|1)}(\lambda_{34})\) is of type 8). Additionally, \(\lambda_{14}\) is 14-atypical. If \(C > 1\), then \(L_{S(3|1)}(\lambda_{14})\) is of type 1). If \(C = 1\), then \(L_{S(3|1)}(\lambda_{14})\) is of type 3).

Assume λ is 24-atypical. Then \(\lambda_{24}\) is 24-atypical. If \(A > 1\), then \(L_{S(3|1)}(\lambda_{24})\) is of type 5) for \(A = 1\), then \(L_{S(3|1)}(\lambda_{24})\) is of type 6).

Lemma 6.6. Assume λ is of type 6). If λ is (14,34)-atypical, then \(N(\lambda)\) is filtered by \(L_{S(3|1)}(\lambda), L_{S(3|1)}(\lambda_{34})\) and \(L_{S(3|1)}(\lambda_{14})\). If λ is 24-atypical, then \(N(\lambda)\) is filtered by \(L_{S(3|1)}(\lambda)\) and \(L_{S(3|1)}(\lambda_{24})\).

Proof. Assume λ is (14,34)-atypical. Then \(\lambda_{34}\) is 34-atypical and \(L_{S(3|1)}(\lambda_{34})\) is of type 8). Additionally, \(\lambda_{14}\) is 14-atypical. If \(C > 1\), then \(L_{S(3|1)}(\lambda_{14})\) is of type 2). If \(C = 1\), then \(L_{S(3|1)}(\lambda_{14})\) is of type 4).
Assume $\lambda$ is 24-atypical. Then $\lambda_{24,34}$ is 34-atypical and $L_{S(3|1)}(\lambda_{24,34})$ is of type 9). \hfill \Box

Lemma 6.7. Assume $\lambda$ is of type 7). If $\lambda$ is $(14,34)$-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda)$, $L_{S(3|1)}(\lambda_{34})$ and $L_{S(3|1)}(\lambda_{14,24})$. If $\lambda$ is 24-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda)$ and $L_{S(3|1)}(\lambda_{24})$.

Proof. Assume $\lambda$ is $(14,34)$-atypical. Then $\lambda_{34}$ is $(24,34)$-atypical and $L_{S(3|1)}(\lambda_{34})$ is of type 10). Additionally, $\lambda_{14,24}$ is 24-atypical. If $A > 1$, then $L_{S(3|1)}(\lambda_{14,24})$ is of type 3). If $A = 1$, then $L_{S(3|1)}(\lambda_{14,24})$ is of type 4).

Assume $\lambda$ is 24-atypical. Then $\lambda_{24}$ is 24-atypical and $L_{S(3|1)}(\lambda_{24})$ is of type 5) for $p \neq 3$ and of type 6) for $p = 3$. \hfill \Box

Lemma 6.8. Assume $\lambda$ is of type 8). If $\lambda$ is 14-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda)$ and $L_{S(3|1)}(\lambda_{14,34})$. If $\lambda$ is 24-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda)$ and $L_{S(3|1)}(\lambda_{34})$.

Proof. Assume $\lambda$ is 14-atypical. If $C > 1$, then $\lambda_{14,34}$ is 34-atypical and $L_{S(3|1)}(\lambda_{14,34})$ is of type 8). If $C = 1$, then $\lambda_{14,34}$ is $(24,34)$-atypical and $L_{S(3|1)}(\lambda_{14,34})$ is of type 10).

Assume $\lambda$ is 24-atypical. If $A > 1$, then $\lambda_{24}$ is 24-atypical and $L_{S(3|1)}(\lambda_{24})$ is of type 8). If $A = 1$, then $\lambda_{24}$ is $(24,34)$-atypical and $L_{S(3|1)}(\lambda_{24})$ is of type 9).

Assume $\lambda$ is 34-atypical. If $C > 1$, then $\lambda_{34}$ is 34-atypical and $L_{S(3|1)}(\lambda_{34})$ is of type 1) if $C = 1$, then $\lambda_{34}$ is $(24,34)$-atypical and $L_{S(3|1)}(\lambda_{34})$ is of type 5). \hfill \Box

Lemma 6.9. Assume $\lambda$ is of type 9). If $\lambda$ is $(14,24)$-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda)$, $L_{S(3|1)}(\lambda_{14})$, $L_{S(3|1)}(\lambda_{24,34})$ and $L_{S(3|1)}(\lambda_{14,34})$. If $\lambda$ is 34-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda)$ and $L_{S(3|1)}(\lambda_{34})$.

Proof. Assume $\lambda$ is $(14,24)$-atypical. Then $\lambda_{14}$ is $(14,34)$-atypical, $\lambda_{24,34}$ is 34-atypical and $\lambda_{14,34}$ is 34-atypical. The module $L_{S(3|1)}(\lambda_{14})$ is of type 6), $L_{S(3|1)}(\lambda_{24,34})$ is a tensor product of a module of type 4) and the Frobenius twist $L_{S(3|1)}^{F}(1,0,0|0)$, and $L_{S(3|1)}(\lambda_{24,34})$ is of type 8).

Assume $\lambda$ is 34-atypical. Then $\lambda_{34}$ is 34-atypical and $L_{S(3|1)}(\lambda_{34})$ is of type 14). \hfill \Box

Lemma 6.10. Assume $\lambda$ is of type 10). If $\lambda$ is 14-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda)$ and $L_{S(3|1)}(\lambda_{14,24})$. If $\lambda$ is $(24,34)$-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda)$, $L_{S(3|1)}(\lambda_{34})$, $L_{S(3|1)}(\lambda_{24})$ and $L_{S(3|1)}(\lambda_{24})$.

Proof. Assume $\lambda$ is 14-atypical. Then $\lambda_{14,24}$ is 24-atypical and $L_{S(3|1)}(\lambda_{14,24})$ is of type 7).

Assume $\lambda$ is $(24,34)$-atypical. Then $\lambda_{34}$ is 34-atypical, $\lambda_{24}$ is 24-atypical and $\lambda_{24}$ is 24-atypical. The module $L_{S(3|1)}(\lambda_{34})$ is a tensor product of a module of type 4) and the Frobenius twist $L_{S(3|1)}^{F}(1,1,0|0)$, $L_{S(3|1)}(\lambda_{24})$ is of type 8), and $L_{S(3|1)}(\lambda_{24})$ is of type 3) for $p \neq 3$ and of type 4) for $p = 3$. \hfill \Box

Lemma 6.11. Assume $\lambda$ is of type 11). If $\lambda$ is $(14,24,34)$-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda)$, $L_{S(3|1)}(\lambda_{34})$, $L_{S(3|1)}(\lambda_{24})$ and $L_{S(3|1)}(\lambda_{14})$. 

Proof. The weight $\lambda_{34}$ is 34-atypical and the module $L_{S(3|1)}(\lambda_{34})$ is a tensor product of the module of type 9) and the Frobenius twist $L_{S(3|1)}^F(1,0,0,0)$. The weight $\lambda_{24}$ is 24-atypical and the module $L_{S(3|1)}(\lambda_{24})$ is a tensor product of the module of type 7) and the Frobenius twist $L_{S(3|1)}^F(1,1,0,0)$. The weight $\lambda_{14}$ is 14-atypical and the module $L_{S(3|1)}(\lambda_{34})$ is of type 12). □

Lemma 6.12. Assume $\lambda$ is of type 12). If $\lambda$ is 14-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda_{34})$ and $L_{S(3|1)}(\lambda_{14})$. If $\lambda$ is (24,34)-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda_{14})$ and $L_{S(3|1)}(\lambda_{24})$ and $L_{S(3|1)}(\lambda_{14},\lambda_{24})$.

Proof. Assume $\lambda$ is 14-atypical. Then $\lambda_{14}$ is 14-atypical. If $C > 1$, then $L_{S(3|1)}(\lambda_{14})$ is of type 13). If $C = 1$, then $L_{S(3|1)}(\lambda_{14})$ is of type 10). Assume $\lambda$ is (24,34)-atypical. Then $\lambda_{34}$ is (14,34)-atypical, $\lambda_{24}$ is (14,24)-atypical and $\lambda_{14,24}$ is 34-atypical. The module $L_{S(3|1)}(\lambda_{34})$ is a tensor product of a module of type 6) and the Frobenius twist $L_{S(3|1)}^F(1,1,0,0)$, $L_{S(3|1)}(\lambda_{24})$ is of type 15), and $L_{S(3|1)}(\lambda_{14,24})$ is of type 4). □

Lemma 6.13. Assume $\lambda$ is of type 13). If $\lambda$ is 14-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda_{34})$ and $L_{S(3|1)}(\lambda_{14})$. If $\lambda$ is (24,34)-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda_{34})$, $L_{S(3|1)}(\lambda_{14})$, $L_{S(3|1)}(\lambda_{24})$ and $L_{S(3|1)}(\lambda_{14},\lambda_{24})$.

Proof. Assume $\lambda$ is 14-atypical. Then $\lambda_{14}$ is 14-atypical. If $C > 1$, then $L_{S(3|1)}(\lambda_{14})$ is of type 13). If $C = 1$, then $L_{S(3|1)}(\lambda_{14})$ is of type 10). Assume $\lambda$ is (24,34)-atypical. Then $\lambda_{34}$ is 34-atypical, $\lambda_{24}$ is 24-atypical and $\lambda_{24}$ is 24-atypical. The module $L_{S(3|1)}(\lambda_{34})$ is a tensor product of a module of type 2) and the Frobenius twist $L_{S(3|1)}^F(1,0,0,0)$, $L_{S(3|1)}(\lambda_{24})$ is of type 1), and $L_{S(3|1)}(\lambda_{24})$ is of type 3) for $C \neq p - 3$ and of type 4) if $C = p - 3$.

Lemma 6.14. Assume $\lambda$ is of type 14). If $\lambda$ is (14,24)-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda)$, $L_{S(3|1)}(\lambda_{24})$, $L_{S(3|1)}(\lambda_{14})$ and $L_{S(3|1)}(\lambda_{14},\lambda_{24})$. If $\lambda$ is 34-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda)$ and $L_{S(3|1)}(\lambda_{34})$.

Proof. Assume $\lambda$ is (14,24)-atypical. Then $\lambda_{24}$ is 24-atypical, $\lambda_{14}$ is 14-atypical and $\lambda_{14}$ is 34-atypical. The module $L_{S(3|1)}(\lambda_{24})$ is a tensor product of a module of type 3) for $A > 1$ or of type 4) for $A = 1$, and the Frobenius twist $L_{S(3|1)}^F(1,0,0,0)$, $L_{S(3|1)}(\lambda_{24})$ is of type 1) for $A > 1$ and of type 8) for $A = 1$. The module $L_{S(3|1)}(\lambda_{14})$ is of type 2).

Assume $\lambda$ is 34-atypical. Then $\lambda_{34}$ is 34-atypical and $L_{S(3|1)}(\lambda_{34})$ is of type 14) for $A \neq p - 3$ and of type 15) for $A = p - 3$.

Lemma 6.15. Assume $\lambda$ is of type 15). If $\lambda$ is (14,24)-atypical, then $N(\lambda)$ is filtered by $L_{S(3|1)}(\lambda)$, $L_{S(3|1)}(\lambda_{24})$, $L_{S(3|1)}(\lambda_{14})$ and $L_{S(3|1)}(\lambda_{14},\lambda_{24})$.

Proof. Assume $\lambda$ is (14,24)-atypical. Then $\lambda_{24}$ is 24-atypical, $\lambda_{14}$ is 14-atypical and $\lambda_{24}$ is 34-atypical. The module $L_{S(3|1)}(\lambda_{24})$ is a tensor product of a module of type 3) for $A > 1$ or of type 4) for $A = 1$, and the Frobenius twist $L_{S(3|1)}^F(1,0,0,0)$, $L_{S(3|1)}(\lambda_{24})$ is of type 1) for $A > 1$ and of type 8) for $A = 1$. The module $L_{S(3|1)}(\lambda_{14})$ is of type 2).

Assume $\lambda$ is 34-atypical. Then $\lambda_{34}$ is (14,24,34)-atypical and $L_{S(3|1)}(\lambda_{34})$ is of type 11).
In the case \( A + C \leq p - 2, A = p - 1, \) or \( C = p - 1, \) the costandard module \( \nabla(\lambda) \) equals \( N(\lambda) \) because \( V = \langle v \rangle. \) Therefore the above 15 lemmas determine all simple composition factors of \( \nabla(\lambda) \) in that case.

If \( A + C > p - 2 \) and \( A, C < p - 1, \) then \( V \) is an extension of \( \langle v \rangle \) by \( \pi, \) and consequently, \( \nabla(\lambda) \) is an extension of \( N(\lambda) \) by \( N(\lambda) = \langle \pi \rangle \oplus (\langle \pi \rangle \otimes Y \oplus \langle \pi \rangle \otimes (Y \wedge Y)). \)

In the case \( A + C > p - 2 \) and \( A, C < p - 1, \) the weight \( \lambda \) is of type 1) or 8) and the simple composition factors of \( N(\lambda) \) are given by Lemmas 6.1 and 6.8. To get the simple composition factors of \( N(\lambda) \), we will explain how to derive the results analogous to Lemmas 6.1 through 6.4 for the highest vector \( \pi. \)

Let \( \lambda = (\lambda_1, \lambda_2, \lambda_3 | \lambda_4) = \lambda + (A + C - p + 2)(\alpha_{12} + \alpha_{23}), \) \( \pi = \lambda_1 - \lambda_2 = p - 2 - A \) and \( \lambda = \lambda_2 - \lambda_3 = p - C. \)

If \( \lambda \) is typical, then \( \lambda \) is atypical. If \( \lambda \) is 14-atypical, then \( \lambda \) is 34-atypical. If \( \lambda \) is 24-atypical, then \( \lambda \) is 24-atypical. If \( \lambda \) is 34-atypical, then \( \lambda \) is 14-atypical. If \( \lambda \) is \( (14, 34) \)-atypical, then \( \lambda \) is \( (14, 34) \)-atypical.

Using these simple observations we can translate the statement of Proposition 5.1 to the case of \( L_{S(3|1)}(\lambda). \) Obviously, only types 1) through 4) are possible for \( L_{S(3|1)}(\lambda). \) Using Lemmas 6.1 though 6.4 we determine the simple composition factors of \( N(\lambda) \) as follows.

**Lemma 6.16.** Assume \( A + C > p - 2 \) and \( A, C < p - 2. \) Then \( \lambda \) is of type 1). If \( \lambda \) is 14-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda_1) \) and \( L_{S(3|1)}(\lambda_3). \) If \( \lambda \) is 24-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda_1) \) and \( L_{S(3|1)}(\lambda_2). \) If \( \lambda \) is 34-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda_1) \) and \( L_{S(3|1)}(\lambda_4). \)

**Lemma 6.17.** Assume \( C = p - 2 \) and \( 0 < A < p - 2. \) Then \( \lambda \) is of type 2). If \( \lambda \) is 14-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda_1) \) and \( L_{S(3|1)}(\lambda_3). \) If \( \lambda \) is 24-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda_1) \) and \( L_{S(3|1)}(\lambda_2). \) If \( \lambda \) is 34-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda_1) \) and \( L_{S(3|1)}(\lambda_4). \)

**Lemma 6.18.** Assume \( A = p - 2 \) and \( 0 < C < p - 2. \) Then \( \lambda \) is of type 3). If \( \lambda \) is 14-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda_1) \) and \( L_{S(3|1)}(\lambda_3). \) If \( \lambda \) is 24-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda_1) \) and \( L_{S(3|1)}(\lambda_2). \) If \( \lambda \) is 34-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda_1) \) and \( L_{S(3|1)}(\lambda_4). \)

**Lemma 6.19.** Assume \( C = p - 2 \) and \( A = p - 2. \) Then \( \lambda \) is of type 4). If \( \lambda \) is 14-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda_1) \) and \( L_{S(3|1)}(\lambda_3). \) If \( \lambda \) is 24-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda_1) \) and \( L_{S(3|1)}(\lambda_2). \) If \( \lambda \) is 34-atypical, then \( N(\lambda) \) is filtered by \( L_{S(3|1)}(\lambda_1) \) and \( L_{S(3|1)}(\lambda_4). \)

Combining Lemmas 6.1, 6.8, 6.16, 6.17, 6.18 and 6.19, we obtain all simple composition factors of \( \nabla(\lambda) \) in the case when \( A + C > p - 2 \) and \( A, C < p - 1. \)

7. Costandard modules for \( S(3|1) \)

Our next goal is to determine the character of a costandard module \( \nabla(\lambda) \) for \( S(3|1). \)

If \( \lambda_3 > 0, \) then all \( w(a, b, c, d, \delta_{14}, \delta_{24}, \delta_{34}) \) are polynomials and \( \nabla(\lambda) = H|_{\lambda}(\lambda). \)

In the case \( \lambda_3 = 0 \) we have \( p_1 \lambda_3. \) The description of the basis and character of a costandard module \( \nabla(\lambda) \) in this case follows from the work of LaScala and Zubkov in [10].
Proposition 7.1. If \( \lambda_3 = 0 \), then \( p|\lambda_4 \), and \( \omega_{34} \equiv 0 \) (mod \( p \)). The character
\[
\chi(\nabla_{S(3|1)}(\lambda))
\]
of the costandard module \( \nabla(\lambda) \) is
\[
\chi(\nabla_{S(3|1)}(\lambda)) = S(\lambda) + \delta_{14} S(\lambda_{24}) + \delta_{12} S(\lambda_{14}) + \delta_{23} S(\lambda_{14,24}).
\]

Proof. Using Theorem 5.4, Proposition 5.6 and Theorem 6.6 of [10] the problem is reduced to a listing of superstandard tableaux that form a basis of \( \nabla(\lambda_1, \lambda_2, 0|0) \).

It follows from [1] that the character of the costandard module \( \nabla(\lambda) \) for hook weight \( \lambda \) is given by the hook Schur function \( HS(\lambda) \) that we represent in the form of a sum \( HS_0(\lambda) + HS_1(\lambda) + HS_2(\lambda) + HS_3(\lambda) \), where
\[
HS_0(\lambda) = x_4^{\lambda_3} P(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3),
\]
\[
HS_1(\lambda) = x_4^{\lambda_3+1} [ \frac{P(\lambda_1 - \lambda_2 - 1, \lambda_2 - \lambda_3, \lambda_3) + P(\lambda_1 - \lambda_2 + 1, \lambda_2 - \lambda_3 - 1, \lambda_3)}{2} + P(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3 - 1, \lambda_3 - 1)],
\]
\[
HS_2(\lambda) = x_4^{\lambda_3+2} [ \frac{P(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3 - 1, \lambda_3) + P(\lambda_1 - \lambda_2 - 1, \lambda_2 - \lambda_3 + 1, \lambda_3 - 1)}{2} + P(\lambda_1 - \lambda_2 + 1, \lambda_2 - \lambda_3, \lambda_3 - 1)],
\]
\[
HS_3(\lambda) = x_4^{\lambda_3+3} P(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 - 1),
\]
where we omit terms where a component of \( P \) is negative.

It follows from Proposition 7.1 that the character of the costandard module \( \nabla(\lambda) \) for non-hook weight \( \lambda \) is given by exactly the same formula. All we have to do is to adopt a convention that terms with negative components in \( P \) are omitted. Therefore we will denote by \( \mathcal{S}(\lambda) \) the Schur function corresponding to the superalgebra \( S(3|1) \) defined as
\[
\mathcal{S}(\lambda) = x_4^{\lambda_3} [S(\lambda_1, \lambda_2, \lambda_3) + S(\lambda_1 - 1, \lambda_2, \lambda_3) x_4 + S(\lambda_1, \lambda_2 - 1, \lambda_3) x_4 + S(\lambda_1, \lambda_2 - \lambda_3, \lambda_3 - 1) x_4^2 + S(\lambda_1, \lambda_2 - \lambda_3 - 1, \lambda_3 - 1) x_4^3],
\]
where \( S(\mu_1, \mu_2, \mu_3) = S_{\mu_1, \mu_2, \mu_3}(x_1, x_2, x_3) \) if \( \mu_1 \geq \mu_2 \geq \mu_3 \geq 0 \) and \( S(\mu_1, \mu_2, \mu_3) = 0 \) otherwise.

This provides a motivation for the general formula describing the character of a costandard module \( \nabla(\lambda) \) for Schur superalgebra \( S(m|1) \), where \( m \geq 1 \), that will be proved in the following section.

Finally, we determine all composition factors of costandard modules \( \nabla(\lambda) \) for a restricted weight \( \lambda \) such that \( \lambda_3 = 0 \). In this case \( \lambda \) is a hook weight and \( \lambda_3 = \lambda_4 = 0 \).

Proposition 7.2. Assume \( \lambda \) is a restricted weight such that \( \lambda_3 = 0 \). If \( \lambda \) is \( 34 \)-atypical, \( (14, 34) \)-atypical or \( (14, 24, 34) \)-atypical, then \( \nabla(\lambda) = L_{S(3|1)}(\lambda) \). If \( \lambda \) is \( (24, 34) \)-atypical, then \( \nabla(\lambda) = L_{S(3|1)}(\lambda_{24}) \).

Proof. Use Propositions 7.1 and 5.1 to verify the equality of characters. If \( \lambda \) is \( (24, 34) \)-atypical, then \( \chi(L_{S(3|1)}(\lambda)) = S(\lambda) + SS(\lambda_{24}) + \delta_{12} S(\lambda_{14}) \) and \( \chi(L_{S(3|1)}(\lambda_{24})) = S(\lambda_{24}) + S(\lambda_{14,24}) \).

The description of composition factors of costandard modules \( \nabla(\lambda) \) for a restricted weight \( \lambda \) is complete.
8. Character of Costandard Modules for $S(m|1)$

The goal of this section is to prove the following character formula for costandard modules of $S(m|1)$ for $m \geq 1$.

Before we can proceed we need to establish a more general notation for Schur superalgebra $S(m|1)$ analogous to Section 1. For more details consult [14].

Let $A = A(m|1)$ be an algebra freely generated over $K$ by elements $c_{ij}$ for $1 \leq i, j \leq m + 1$, where elements $c_{ij}$ for $1 \leq i, j \leq m$ and $c_{m+1,m+1}$ are even, and $c_{i,m+1}, c_{m+1,i}$ for $1 \leq m$ are odd. Denote $\mathfrak{c} = (c_{ij})_{1 \leq i,j \leq m}$, $D = \text{det}(\mathfrak{c})$ and $\text{Adj}(\mathfrak{c}) = (A_{ij})_{1 \leq i,j \leq m}$, the adjoint matrix of the matrix $\mathfrak{c}$.

The localization of $A(m|1)$ by $D$ and $c_{m+1,m+1}$ is the coordinate algebra $K[G]$ of the general linear supergroup $G = GL(m|1)$.

The $G$-supermodule $H^0_G(\lambda)$ can be described using an explicit isomorphism $\phi : H^0_G(\lambda) \otimes K[c_{1,m+1}, \ldots, c_{m,m+1}] \rightarrow H^0_G(\lambda)$ defined in Lemma 5.2 and on p.163 of [10]. Isomorphism $\phi$ acts as an identity on elements of $c_{ij}$ for $1 \leq i, j \leq m$ and is given as

$$\phi(c_{i,m+1}) = A_{i1}c_{1,m+1} + \cdots + A_{im}c_{m,m+1} = y_{i,m+1}$$

for $1 \leq i \leq m$ and

$$\phi(c_{m+1,m+1}) = c_{m+1,m+1} - c_{m+1,1}y_{1,m+1} - \cdots - c_{m+1,m}y_{m,m+1} = z.$$

Lemma 8.1. Let $\delta_{i,m+1} \in \{0, 1\}$ for $1 \leq i \leq m$ and $T \subset \{1, \ldots, m\}$. Then

$$D \prod_{i \in T} y_{i,m+1} \in A(m|n).$$

Proof. List the elements of $T$ as $i_1 < \ldots < i_k$. Then

$$\prod_{i \in T} D y_{i,m+1} = \prod_{i=1}^{k} (A_{i1}c_{1,m+1} + \cdots + A_{im}c_{m,m+1}) = \sum \sigma A_{i1,\sigma_1} \cdots A_{ik,\sigma_k} c_{\sigma_1,1,m+1} \cdots c_{\sigma_k,m+1}$$

where $\sigma$ runs through all permutations of $k$ elements from the set $\{1, \ldots, m\}$. The last expression equals

$$\sum_{1 \leq j_1 < \cdots < j_k \leq m \mu \in \Sigma_k} (-1)^\mu A_{i_1,j_{\mu(1)}} \cdots A_{i_k,j_{\mu(k)}} c_{j_1,j_1} \cdots c_{j_k,m+1}.$$

Since each term $\sum_{\mu \in \Sigma_k} (-1)^\mu A_{i_{\mu(1)},j_{\mu(1)}} \cdots A_{i_{\mu(k)},j_{\mu(k)}}$ is the minor of the adjoint matrix $\text{Adj}(\mathfrak{c})$, the Jacobi Theorem on minors of the adjoint matrix (See [3], p.57 or Theorem 2.5.2 of [13]) states that it is a product of its complementary minor of size $m-k$, which is a polynomial element, and of $D^{k-1}$. Therefore $D \prod_{i \in T} y_{i,m+1}$ is a polynomial element. \qed

Proposition 8.2. If $\lambda$ is a hook weight of $G$, that is if $\lambda_m \geq 1$, then $H^0_G(\lambda) = \nabla(\lambda)$.

Proof. If $\lambda$ is a polynomial weight, then each element of $H^0_G(\lambda)$ can be written as a product of the following elements: $D^\lambda$, a polynomial element from $K[c_{ij}, 1 \leq i, j \leq m]$, $z^{\lambda_{m+1}}$ and a product of some pairwise different elements $y_{i,m+1}$. Since $\lambda_m \geq 1$, Lemma 8.1 implies that each such element is polynomial. \qed

Remark 1. An analogous argument proves that $H^0_G(\lambda) = \nabla(\lambda)$ for hook weights $\lambda$ of $GL(m|n)$. In fact, there is an unpublished result of Alexander N. Zubkov stating that $H^0(\lambda) = \nabla(\lambda)$ if and only if $\lambda_m \geq n$.

Now we can prove the character formula for $\nabla(\lambda)$.
Theorem 8.3. The character of costandard module $\nabla(\lambda)$ for Schur superalgebra $S(m|1)$ is given by the formula

$$\mathcal{S}_\lambda(x_1, \ldots, x_m|x_{m+1}) = x_{m+1} \sum_{\delta_i = (\delta_{i,m+1})} S(\lambda_1 - \delta_{1,m+1}, \ldots, \lambda_{m-1} - \delta_{m-1,m+1}, \lambda_m - \delta_{m,m+1}) x_{m+1} \prod_{i \in \{1, \ldots, m\}} \delta_{i,m+1},$$

where $S(\mu_1, \ldots, \mu_m) = S_{\mu_1, \ldots, \mu_m}(x_1, \ldots, x_m)$ is the Schur function if $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0$ and $S(\mu_1, \ldots, \mu_m) = 0$ otherwise.

Proof. If $\lambda_m \geq 1$, then $\lambda$ is a hook weight of $G$, and Proposition 8.2 implies that the character of $\nabla$ is given as the hook Schur function $HS(\lambda)$. Part 6.3 of [1] shows that the length of each row in a skew diagram appearing in the definition of the hook Schur function does not exceed one. Therefore the formula in our theorem describes the function $HS(\lambda)$.

If $\lambda_m = 0$, then Theorem 5.4 and Proposition 5.6 of [10] reduce our problem to the computation of the character of the costandard module $\nabla(\lambda^+|0)$. An explicit basis of this module, consisting of superbiddeterminants corresponding to superstandard tableaux, is given by by Theorem 6.6 of [10]. After we write down all superstandard tableaux corresponding to this module, the statement of the theorem follows easily.

9. Corrections to the description of costandard modules for Schur superalgebra $S(2|1)$ in [6]

Statement of Proposition 1.1 of [6] is valid for the restricted weights $\lambda$ of $S(2|1)$ only. The dimension and the character formula of the simple module $L_\lambda$ for $S(2|1)$ can be derived from this proposition using the Steinberg tensor product theorem from [9].

The statement of Proposition 2.1 of [6] should be modified as follows:

1) Only the restricted weight $\lambda$ should be considered.

2) If the weight $\mu$ of a simple module $L_\mu$ listed in the filtration of the costandard module $\nabla(\lambda)$ is not dominant, then the simple module $L_\mu$ should be omitted from the filtration of $\nabla(\lambda)$.

References


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