Viscous fluid flow inside an oscillating cylinder and its extension to Stokes’ second problem

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We present the analytical solution for the fluid motion inside a cylindrical tank whose angular velocity starts from rest and undergoes a harmonic oscillation. This problem, which has not yet been reported, is an extension to Stokes’ second problem where the fluid motion is governed by an outer moving cylindrical boundary and a zero velocity condition at the cylinder center. Different from the flow on the outside of a cylinder, the cylinder radius has a large influence on the internal fluid motion. We show that the fluid approaches solid body rotation for cylinders with outer radii similar to the characteristic viscous length-scale of the flow, whereas the motion approaches that of Stokes’ original flat plate solution within very large cylinders. We detail both the transient starting condition and the quasi-steady fluid motion, which we present along with a particle image velocimetry experiment for validation. After decay of the initial startup transient, both quasi-steady analytical and experimental results predict that the oscillatory flow inside has an amplitude of velocity that decreases towards the center of the cylinder. The thickness of the Stokes layer, which is proportional to the penetration depth of the viscous wave, is altered by the size of the cylinder and/or the frequency of oscillation. We show that the penetration depth of the Stokes layer reaches its maximum thickness at intermediate cylinder sizes. The solution and results presented herein are potentially of value to describe the fluid motion in many applications where fluids are contained within cylindrical geometries.

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I. INTRODUCTION

Stokes’ second problem is one of the famous exact solutions to the Navier-Stokes equations describing oscillatory flows.\(^1\) Originally analyzed by G.G. Stokes\(^2\) and later by Lord Rayleigh,\(^3\) the problem is one in which an infinitely long flat wall oscillates harmonically in a fluid. Because of the no-slip boundary condition and viscosity, the fluid near the wall follows a harmonic motion that is parallel to the wall. As a result, the velocity profile of the fluid is similar to that of a damped harmonic oscillation, where layers within the fluid can be identified based on their phase lag with the wall oscillation.

Since the original quasi-steady solution, this laminar oscillating flow has been described by only a few additional analytical investigations. Erdogan\(^4\) provided an exact solution for an infinite flat plate considering a stationary start where the wall motion started from rest. His result included a transient term and a quasi-steady term. The transient term decayed rapidly within a few cycles of boundary oscillations and had the same expression of quasi-steady state that was originally developed by Stokes. Li and Marfatia\(^5\) provided the expression for Stokes’ second problem for flow on the outside of a cylinder with a harmonically oscillating rotation. Rivero et al.\(^6\) later analyzed this analytical solution and conducted experiments to validate their results. Both results consisted of a transient solution for a stationary starting condition and a quasi-steady solution similar to Stokes’ classical formulation. They showed that the dimensionless quasi-steady oscillatory flow on the outside of a cylinder was smaller in magnitude compared to a flat plate when the cylinder was small, but that the solution converged rapidly to the flat plate solution in both transient and quasi-steady terms as the radius of the cylinder increased. Von Kerczek and Davis\(^7\) and Hall and Stuart\(^8\) have also used linear stability theory to predict the limiting conditions for these quasi-steady analytical solutions for flat plate and external cylindrical geometries, respectively.

Due to its applicability to only laminar flows with an oscillating boundary, Stokes’ second problem has seen limited application outside of the above-mentioned studies. Ai and Vafai\(^9\) theoretically extended Stokes’ second problem to three different types of non-Newtonian fluids in contact with an oscillating flat plate. Their results showed that the magnitude of the fluid velocity was highly dependent on the non-Newtonian fluid behavior, which they captured using appropriate models. Flows generated by an oscillating wall have also had applications related to plasma actuators, where Hehner et al.\(^10\) developed a novel oscillatory plasma actuator by forming a Stokes layer. Their results indicated a potential application to turbulent drag reduction.
Fluids in contact with the inner surface of a moving cylinder, for example liquids in some coating processes (for example see Refs. 11 and 12) or water rolled for particle content analysis in cylindrical tanks (see Ref. 13), offer a unique potential application of flows with Stokes layers. In this geometry, the fluid is contained inside of an outer moving cylindrical wall. The analytical solution for the fluid motion subject to an outer cylindrical boundary oscillation, which is opposite the problem solved by Li and Marfatia and Rivero et al., is currently absent from the literature. In this paper, we mathematically analyze the startup transient and quasi-steady fluid motion inside of a completely full cylinder and validate our solution experimentally. The paper is organized into five subsequent sections. In section II, we formulate the analytical solution of the fluid velocity profile in cylindrical coordinates. In section III, we describe the implementation of a particle image velocimetry (PIV) experiment with an optically transparent oscillating cylindrical tank and, in section IV, present the results of our validation study for the quasi-steady state. In section V, we discuss the unique effect of the dimensionless parameters on the flow and, finally, we make concluding remarks in section VI.

II. MATHEMATICAL FORMULATION

In this problem, we assume an incompressible Newtonian fluid with a constant kinematic viscosity $\nu$ and density $\rho$ inside an infinitely long cylinder of radius $R$. We neglect pressure and gravity forces and assume that the cylinder oscillates around its center at low-enough speeds for the flow to remain laminar. Fig. (1) displays a schematic diagram of this flow, where all fluid motion is in the azimuthal direction and we assume that the radial and axial components of velocity can be neglected. We can, thus, reduce the Navier-Stokes equation in cylindrical coordinates to
\[
\frac{\partial u_\theta}{\partial t} = \nu \left( \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{\partial^2 u_\theta}{\partial r^2} - \frac{u_\theta}{r^2} \right),
\]

where \( u_\theta \) is the azimuthal velocity, \( t \) is time, and \( r \) is the radial position. Due to the geometry, the azimuthal velocity magnitude at the center of the cylinder must be zero. We can thus solve Eq. (1) subject to the boundary conditions that include this bounded center and the velocity boundary condition at the wall. For our harmonically oscillating cylinder, we represented the boundary and initial conditions as

\[
\begin{align*}
\theta &= a \cdot \cos(\omega t) \quad \text{at} \ r = R, \ t \geq 0, \\
\theta &= 0 \quad \text{at} \ r = 0, \\
\theta &= 0 \quad \text{at} \ t < 0.
\end{align*}
\]

Here, \( a \) and \( \omega \) are the amplitude and frequency of oscillation. The cylinder begins to oscillate about its axis at time \( t = 0 \). We use a cosine function to represent the harmonic oscillation. Though they differ in their startup transient, either a sine or cosine function lead to the same quasi-steady behavior. We describe the solution procedure in more detail below, with the solution for a sine function as the boundary condition provided in Appendix A.

A. Analytical solution

We can non-dimensionalize Eq. (1) by defining a dimensionless velocity \( f \), time \( \tau \) and length \( \eta \) as follows,

\[
\begin{align*}
f &= \frac{u_\theta}{a}, \\
\tau &= \omega t, \\
\eta &= \frac{r}{\delta},
\end{align*}
\]

where \( f \) is the ratio of the azimuthal velocity to the maximum velocity at the wall and \( \eta \) is the radial position with respect to the characteristic lengthscale \( \delta \). For oscillating flows, the oscillation frequency \( \omega \) and the viscosity of the fluid \( \nu \) usually determine the characteristic thickness of the oscillatory boundary layer. Therefore we choose \( \delta \) as
\[ \delta = \sqrt{\frac{v}{\omega}}, \]  

(4)

which has the same definition as that used by previous solutions.\(^4\) Next, we can non-dimensionally represent the one-dimensional oscillatory flow inside a cylinder as

\[ \frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial f}{\partial \eta} - \frac{f}{\eta^2}, \]  

(5)

\[ f = \cos(\tau) \text{ at } \eta = \eta_R, \quad \tau \geq 0, \]  

(6a)

\[ f = 0 \text{ at } \eta = 0, \]  

(6b)

\[ f = 0 \text{ at } \tau < 0. \]  

(6c)

where \( \eta_R = R/\delta \) is a constant representing the dimensionless outer radius of the tank. From here, we apply the Laplace transform to solve the initial value problem, \( L[f(\eta, \tau)] = F(\eta, s) \).\(^4\) We can then write the governing equation after the Laplace transformation with a zero initial condition as

\[ \eta^2 \frac{d^2 F}{d\eta^2} + \eta \frac{dF}{d\eta} - (s\eta^2 + 1)F = 0, \]  

(7)

where \( s \) is a constant. The two boundary conditions are now defined as

\[ F(\eta, s) = \frac{s}{s^2 + 1} \text{ at } \eta = \eta_R, \]  

(8a)

\[ F(\eta, s) = 0 \text{ at } \eta = 0. \]  

(8b)

Eq. (7) is the modified Bessel differential equation\(^5\) and its general solution is given as

\[ F(\eta, s) = C_1 I_1(\eta \sqrt{s}) + C_2 K_1(\eta \sqrt{s}), \]  

(9)

where \( I_1 \) is the modified Bessel function of the first kind and of order one and \( K_1 \) is the modified Bessel function of the second kind and of order one. By applying the boundary conditions of Eqs. (8), we can solve for the two constants \( C_1 \) and \( C_2 \) as

\[ C_1 = \frac{s}{s^2 + 1} \frac{1}{I_1(\eta_R \sqrt{s})}. \]  

(10a)
\[ C_2 = 0. \] (10b)

Therefore, the general solution of Eq. (9) becomes

\[ F(\eta, \tau) = \frac{s}{s^2 + 1} \cdot \frac{I_1(\eta \sqrt{s})}{I_1(\eta_R \sqrt{s})}, \] (11)

Here we can obtain a solution through the use of the inverse Laplace transform, which defines the dimensionless velocity \( f \) as

\[ f(\eta, \tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s}{s^2 + 1} \cdot \frac{I_1(\eta \sqrt{s})}{I_1(\eta_R \sqrt{s})} \cdot \exp[s \tau] ds. \] (12)

We can then solve Eq. (12) to find the dimensionless solution for the azimuthal velocity inside of our cylinder. Eq. (12) is similar in form to solutions provided for Stokes’ second problem on the outside of a cylinder,\(^5\) but differs in that their solution used \( K_1 \) instead. The Bessel function \( K_1(z) \) converges to 0 as \( z \) increases, which is appropriate for the far-field boundary condition of an external flow around a cylinder in an infinite stagnant fluid. To solve the inverse Laplace transform in Eq. (12), we can make use of the Residue Theorem. We start by defining a function \( g(s) \) as the integrand of Eq. (12),

\[ g(s) = \frac{s}{s^2 + 1} \cdot \frac{I_1(\eta \sqrt{s})}{I_1(\eta_R \sqrt{s})} \cdot \exp[s \tau], \] (13)

where \( s \) can be any complex number. The terms in the denominator of Eq. (13) indicate \( g(s) \) has simple poles at \( \pm i \) and at all values where \( I_1(\eta_R \sqrt{s}) = 0 \), where \( z_0 \) is a simple pole of the complex-value function \( g \), expressed as

\[ \text{Res}[g,z_0] = \lim_{s \to z_0} (s - z_0) g(s). \] (14)

For the simple poles at \( z_0 = \pm i \), the two residues are

\[ \text{Res}[g,i] = \frac{1}{2} \exp[i \tau] \frac{I_1(\eta \sqrt{i})}{I_1(\eta_R \sqrt{i})}, \] \hspace{2cm} (15a)

\[ \text{Res}[g,-i] = \frac{1}{2} \exp[-i \tau] \frac{I_1(\eta \sqrt{-i})}{I_1(\eta_R \sqrt{-i})}. \] \hspace{2cm} (15b)
To define the residues for the poles associated with $I_1(\eta_R \sqrt{s}) = 0$, we need to find all zeros for this function in the complex plane. $I_1(z)$ has only one real zero when evaluated at $z = 0$ and we can find the residue for the resulting pole at $z_0 = 0$ through Eq. (14) and use of L’Hopital’s rule,$^{15}$

$$\text{Res}[g,0] = \lim_{s \to 0} (s-0) \frac{s}{s^2 + 1} \cdot \frac{I_1(\eta \sqrt{s})}{I_1(\eta_R \sqrt{s})} \cdot \exp[s \tau] = 0. \quad (16)$$

We can find imaginary zeros of $I_1$ by substituting $I_1(iz) = iJ_1(z)$, where $J_1(z)$ is the Bessel function of the first kind and of order one. All zeros of the odd function $J_1(z)$ occur for simple and real values of $z$.$^{15}$ If we define $z = \pm \alpha_n$ to be the $n^{th}$ zero of $J_1(z)$, and $s = z_0,n$ as the $n^{th}$ zero of $I_1(\eta_R \sqrt{s})$, we have $i\eta_R \sqrt{z_0,n} = \pm \alpha_n$, which is equivalent to

$$z_{0,n} = \frac{\alpha_n^2}{\eta_R^2}. \quad (17)$$

Eq. (17) is, thus, an expression for the remaining simple poles, $z_{0,n}$, of $g(s)$. Next, we can re-write Eq. (13) into the following form,

$$g(s) = \frac{p(s)}{q(s)}, \quad (18)$$

where

$$p(s) = \frac{s}{s^2 + 1} \cdot I_1(\eta \sqrt{s}) \exp[s \tau], \quad (19)$$

$$q(s) = I_1(\eta_R \sqrt{s}). \quad (20)$$

Corollary 22.1 on page 735 of O’Neil$^{14}$ shows that, if $q(s)$ in Eq. (18) has a simple zero at $z_0$ and $p(z_0) \neq 0$, the residue of $g(s)$ at the resulting pole can be written as

$$\text{Res}[g,z_{0,n}] = \frac{p(z_{0,n})}{q'(z_{0,n})} \quad (21)$$

$$= \exp[z_{0,n} \tau] \cdot \frac{4z_{0,n}^{3/2}}{(z_{0,n}^2 + 1) \eta_R} \cdot \frac{I_1(\eta \sqrt{z_{0,n}})}{I_0(\eta_R \sqrt{z_{0,n}}) + I_2(\eta_R \sqrt{z_{0,n}})},$$

where $q'$ is the derivative of $q$ with respect to $s$ and $I_0$ and $I_2$ are modified Bessel functions of the first kind and of order zero and two, respectively. Taking the sum of Eqs. (15a), (15b), (16) and (21), we can arrive at the expression of the solution to Eq. (12) as
Eq. (22) is, thus, the analytical solution for the dimensionless velocity $f$ for one-dimensional laminar fluid flow inside of a harmonically oscillating cylindrical wall. Eq. (22) consists of two parts. The first part is the solution for the quasi-steady dimensionless velocity, $f_s$, which has a period of $2\pi$ and can be written as

$$f_s(\eta, \tau) = \frac{1}{2} \left( \exp[i\tau] \frac{I_1(\eta \sqrt{i})}{I_1(\eta_R \sqrt{i})} + \exp[-i\tau] \frac{I_1(\eta \sqrt{-i})}{I_1(\eta_R \sqrt{-i})} \right).$$

The infinite summation in Eq. (22), therefore, is necessary to describe the transient starting condition of the fluid motion. This second part of the solution decays rapidly with time $\tau$ due to the exponential function of the negative $z_{0,n}$. 

$$f_s(\eta, \tau) = \frac{1}{2} \left( \exp[i\tau] \frac{I_1(\eta \sqrt{i})}{I_1(\eta_R \sqrt{i})} + \exp[-i\tau] \frac{I_1(\eta \sqrt{-i})}{I_1(\eta_R \sqrt{-i})} \right).$$
III. EXPERIMENTAL METHODS

A. Experimental facility

To verify the validity of quasi-steady motion expressed in Eq. (23), we performed PIV experiments of this internal cylindrical oscillating flow. Song and Rau\textsuperscript{16} originally developed the experimental facility, which consisted of a transparent cylindrical acrylic tank that rested on four rollers, as shown schematically in Fig. 2 (a). A programmable electric servo motor (Dynamixel MX-28T, Trossen Robotics) connected to one end of the tank with a belt-pulley system controlled the rotational motion of this tank. The tank was 300 mm in length, 100 mm in inner diameter with a wall thickness of 5 mm. We determined the tank was long enough that we could neglect the effect of secondary flows on measurements acquired along the central cross-section of the tank by estimating the end-wall boundary layer thickness.\textsuperscript{17,18} We precisely controlled the servo motor using a U2D2 controller (Trossen Robotics).

We illuminated the PIV flow tracers using a continuous laser with a wavelength of 532 nm (Opus 532, Laser Quantum) which was formed into a light sheet that bisected the tank using a Powell lens (75°, Edmund Optics). The light sheet was parallel to the end caps of the tank so that the azimuthal fluid velocity could be captured by the illuminated flow tracers. An example PIV image obtained with this illumination scheme is shown in Fig. 2 (b).

We mounted a high-speed camera (FASTCAM Mini AX200, Photron) such that its viewing axis was perpendicular to the illuminated PIV particles, which it viewed through the transparent end cap of the cylindrical tank. We used a macro lens with a 200 mm focal length (AF Micro-Nikkor 200mm f/4D IF-ED) to focus the camera on the light sheet. This lens provided a field of view (FOV) roughly 28 mm × 28 mm (27 μm per pixel resolution with the 1024 × 1024 pixel sensor of the camera). Because this FOV was not big enough to view the entire tank cross section, we positioned the camera such that the fluid near the bottom tank wall was in view, as shown in Fig. 2 (b). To reduce the effects of wall reflections to enable near-wall measurements, we seeded the water with red fluorescent polystyrene PIV particles (PSFR010UM, MagSphere) and installed a wavelength filter (Orange M62.0 × 0.75 filter, ThorLabs) on the end of the camera lens. We calculated the seeding density from our experimental images to be on average around 15 particles per 32 × 32 pixel window.

We recorded PIV image frames as a time series at a rate of 500 fps with a shutter speed of...
FIG. 3. The center of the cylindrical tank as determined from fitting a circle to the tank wall within the camera FOV.

1/1000 s. The period for one oscillation of this quasi-steady problem depends on the oscillation frequency $\omega$, as given in Eq. (2a). We recorded data for each test for at least one period of oscillation after a long enough time had passed to avoid the startup transient.

B. Data processing

We calculated velocity vectors from our PIV images using codes developed by Eckstein and Vlachos$^{19,20,21}$ and available online.$^{22}$ Before PIV analysis, we calculated the local minimum intensity at each pixel from the entire image dataset and subtracted this intensity from each image. This process removed all background intensity that remained stationary throughout the experiment. We implemented three passes for the velocity evaluation. The first pass used a square evaluation window with a size of $32 \times 32$ $(x \times y)$ pixels and a window overlap of 50% in both $x$ and $y$ directions. After the initial pass, we reduced the window size to $32 \times 8$ pixels with a window overlap of 75%. We reduced the window height to eight pixels to better resolve the velocity profile in the boundary layer,$^{23}$ which was predominantly in the $y$-direction given the placement of our FOV within this cylindrical geometry. Overall, this evaluation scheme gave us vector spacing of 5
vectors per mm in the $x$-direction and 18 vectors per mm in the $y$-direction. We applied a universal outlier detection (UOD)$^{24}$ after each pass to eliminate bad vectors and a Gaussian smoothing filter after the first and second passes. We used the robust phase correlation,$^{19–21}$ with iterative image deformation to compensate for in-plane velocity gradients.$^{25}$

The camera viewed a portion of the tank wall during the experiments, which was outlined by a small amount of PIV particles that had adhered to the wall during the initial test setup. This outline can be seen in Fig. 2 (b), where the internal wall boundary appears as a uniform bright curve. Because the camera FOV did not include the center of the tank, we calculated the location of the tank center by fitting a circle to this illuminated wall arc. This fitting process is shown schematically in Fig. 3. The radial locations of the fluid within the FOV could then be determined from their locations relative to the wall boundary and the tank center.

To obtain azimuthal velocity profiles, we first linearly interpolated the measured velocity vectors (in Cartesian coordinates from our PIV analysis) to a circular grid with a radial spacing of 5.9 vectors per mm. We could then calculate the spatially-averaged azimuthal velocity at each radial location from the interpolated vectors at each experimental time step. This azimuthal averaging reduced our measurement uncertainty, discussed below, and is consistent with the one-dimensionality of our flow geometry.

IV. RESULTS

A. Analytical solution

Fig. 4 (a) shows the analytical velocity profiles considering only the quasi-steady solution, calculated with Eq. (23), for an infinitely long cylinder with a dimensionless radius of $\eta_R = 50$. In this figure, $\tau = 0$ corresponds to the beginning of the cosine boundary oscillation. Here, we plot the instantaneous velocity at six different phase angles during one period of oscillation. As expected, the amplitude of dimensionless velocity is high near the wall and becomes smaller as $\eta$ decreases. Though not shown in the plot, the velocity becomes zero at $\eta = 0$. We determine a positive velocity envelope from the norm of the quasi-steady velocity in Eq. (23). With a symmetric negative envelope, they represent the greatest magnitude of velocity over time as a function of $\eta$. The distance between envelopes can be used to determine $\delta_s$, which is the penetration depth of the viscous wave and is defined as the point at which the value of the envelope magnitude reaches
FIG. 4. (a) One period of quasi-steady velocity profiles plotted versus dimensionless radius during one
oscillation of the cylinder (cylinder radius is $\eta_R = 50$), and (b) quasi-steady dimensionless velocity versus
dimensionless time at the cylinder boundary and three radial distances within the fluid.

0.01. The case shown in Fig. 4 ($\eta_R = 50$) has a penetration depth $\delta_s = 6.6\delta$. This value is slightly
greater than that of the penetration depth of Stokes’ second problem for a flat plate, given as $6.5\delta$. The
thicker layer in the present problem can be attributed to the cylindrical geometry, as we will
discuss in greater detail below. Fig. 4 (b) shows the velocity boundary condition at $\eta = \eta_R$, given
by Eq. (2a), plotted as a function of dimensionless time. In this figure we also plot the time-varying
quasi-steady velocity at three locations close to the boundary; $\eta = 47$, 48, and 49. The velocity of
the fluid inside the tank follows the motion of the boundary oscillation but with reduced amplitude
and a lag in phase. At a dimensionless radius of $\eta = 47$, the maximum velocity is 12.4% of that at
the boundary.

Fig. 5 shows the fluid velocity profiles considering the transient start, Eq. (22), at early dimen-
sionless times for the same boundary condition and cylinder with $\eta_R = 50$. We plot in total four
velocity profiles within a quarter of the first oscillation period, along with one instant of the quasi-
steady velocity profile. The first 500 terms of the infinite series are used in Fig. 5 and the truncation
error is estimated to be $O(10^{-15})$. Fig. 5 shows how the transient motion from the initial startup
vanishes as time increases and the velocity profiles approach their quasi-steady values. After one
quarter oscillation ($\tau = \pi/2$), the maximum difference in dimensionless velocity compared to the
quasi-steady velocity at the corresponding phase angle reduces to less than 0.06. After the first
full cycle of oscillation, the difference is less than 0.01.
FIG. 5. Transient velocity profiles, Eq. (22), at different dimensionless times plotted versus dimensionless radius. The solid line with circles represents the quasi-steady velocity profile, Eq. (23), at a phase angle of $\pi/2$. The first 500 positive zeros of $J_1(z)$ are used for Eq. (22).

B. Experimental validation

We implemented a series of PIV measurements using the experimental facility described in section III to validate the analytical solution of quasi-steady motion given by Eq. (23). The oscillation frequency of the cylindrical tank was 0.5 rad/s and oscillation magnitude was 9 cm/s, giving us the boundary condition of

$$u_\theta(R,t) = 9 \cdot \cos(0.5t).$$  \hspace{1cm} (24)

The kinematic viscosity $\nu$ of the water (20.6°C) was $9.8 \cdot 10^{-7}$ m²/s and the radius of the cylinder was $R = 5$ cm, which resulted in a dimensionless radius of $\eta_R = 35.7$ for this test case. Fig. 6 (a) shows dimensional velocity profiles plotted versus radius at two distinct times during the oscillation, where $t = 0$ s was the starting time of the experimental recording. The experimental results closely follow the analytical solution and display the boundary layer clearly near the tank wall. Fig. 6 (b) shows the velocity variation over time at a fixed radial distance of $r = 4.77$ cm from the center of the tank, which also closely follows the analytical solution given by Eq. (23). We analyzed the difference between the analytical and experimental results and the time-average root-mean-square deviation was 0.09 cm/s, with a maximum of 0.22 cm/s at $r = 4.97$ cm. Uncertainty in the fluid viscosity, which was calculated for water at room temperature, and the synchronization between the analytical and experimental results could account for this deviation. Based on the
FIG. 6. PIV validation of our quasi-steady analytical solution for a tank with dimensionless radius of \( \eta_R = 35.7 \) (\( R = 5 \) cm, \( \omega = 0.5 \) rad/s) showing, (a) comparisons between the quasi-steady analytical solution and PIV results at two time instants and, (b) velocity variation over one period at a radial distance \( r = 4.77 \) cm.

resolution of our measurements, we estimate that the results could be out-of-sync by a maximum of 0.02 s, which would result in a velocity discrepancy of up to 1.9%. It should be noted that we observed no flow instabilities in the experimental results at these conditions and that the flow was stable and repeatable over many oscillations.

We analyzed the uncertainty in our PIV measurements following the method developed by Xue, Charonko, and Vlachos,\textsuperscript{26} which utilizes the mutual information between particle image patterns to estimate uncertainty. Effectively, this method estimates uncertainty based on the strength of the PIV correlation. We present velocity uncertainty as the vertical error bars shown in both Fig. 6 (a) and (b). Uncertainty in our measurements were highest in locations very near the wall (\( r > 4.92 \) cm) since there were fewer PIV particles in this region and, thus, poorer PIV correlation. The maximum root-sum-square uncertainty of velocity in this near-wall region was 7.8%, compared to the oscillation magnitude at the wall. The locations further from the wall had uncertainty in velocity of 2% or less. In addition to the velocity determination, there were also uncertainties in our calculation of radial distance from the center of the tank. During our analysis, we observed that the inner wall of the tank moved vertically within the camera FOV by a maximum of 25 pixels (0.7 mm) over each rotation, which we determined was caused by a non-constant tank wall thickness. The fitting process used to find the tank center (described in section III) accounted for this displacement, but did not eliminate the position uncertainty. We estimated the uncertainty
in the radial position to be 0.12 mm based on the fitted results. Given the excellent agreement between our analytical solution and experimental results, the tank imperfection did not appear to significantly affect the quasi-steady azimuthal fluid velocities in the tank.

V. DISCUSSION

This problem can be thought of as an extension to the classic Stokes’ second problem, where there now exists a balance between the penetration depth of the viscous effects and the finite radius of the cylinder. From the analytical solution of quasi-steady dimensionless velocity, the velocity profile given by Eq. (23) is only affected by the size of the cylinder $R$, the oscillation frequency $\omega$, and the viscosity of the fluid $\nu$, all of which are captured in the nondimensional radial position $\eta$. Based on this non-dimensional relationship, we can expect that decreasing the size of cylinder will have a similar effect on the dimensionless velocity profile as decreasing the oscillation frequency or using a more viscous fluid. At the same time, the amplitude of the oscillation $a$, which only determines the magnitude of the azimuthal velocities, will not influence the penetration depth per radius as long as the flow remains laminar. In Fig. 7, we plot velocity profiles for four different $\eta_R$, which show how the fluid velocity profile penetrates further into the cylinder as $\eta_R$ decreases. This plot indicates that thick Stokes layers, relative to the cylinder radius, can be generated by using small cylinders, low-frequency oscillations, or more-viscous fluids.

Unlike Stokes’ second problem of external flows on an infinite flat plate\textsuperscript{1–3} or outside a
which have a single boundary, our solution for the flow inside a cylinder is bounded by the cylinder center and the cylinder outer wall. This leads to two distinct solutions at the limit of very small and large cylinders. When the dimensionless cylinder size $\eta_R$ approaches infinity, the penetration depth thickness approaches that of Stokes’ second problem for a flat plate, as shown in Fig. 7 (b). As we decrease the cylinder radius, the penetration depth per radius initially thickens relative to $\delta$, but then quickly decreases. As the cylinder size approaches $\delta$ ($\eta_R = 1$), the slope of $\delta_s/\delta$ as a function of $\eta_R$ converges to a value of 0.99, which is equivalent to $\delta_s = 0.99R$. Given that the definition of $\delta_s$ is the point at which the velocity envelope decays to 1% of the boundary velocity amplitude, this equality is consistent with solid body rotation. Recalling that the characteristic lengthscale of this problem is defined as $\delta = \sqrt{\nu/\omega}$, we can define the characteristic velocity as the oscillation magnitude $a$, and the Reynolds number for the problem as $Re = a\delta/\nu = a/\sqrt{\nu \omega}$, which has the same expression as that used by previous studies.\(^6\) We can also define a Strouhal number based on the oscillation frequency of $2\pi \omega$, where $St = 2\pi \omega R/a$. Multiplying the Reynolds number by the Strouhal number we obtain,

$$Re \cdot St = \frac{a}{\sqrt{\nu \omega}} \cdot \frac{2\pi \omega R}{a} = \frac{2\pi R}{\sqrt{\nu \omega}} = 2\pi \eta_R. \quad (25)$$

The product of the two dimensionless numbers is, thus $2\pi$ times the dimensionless radius $\eta_R$. This value indicates the ratio of oscillatory momentum to the viscous effects. Small cylinder sizes are analogous to a small value of $Re \cdot St$ and indicate that viscous forces dominate the flow. Under these conditions, near-linear velocity profiles result from the solution as shown in Fig. 7 (a) for $\eta_R = 1$. From this, we can conclude that Stokes flow is maintained for dimensionless cylinder radii $\eta_R \ll 1$.

VI. CONCLUSION

We have developed the analytical solution for the oscillatory flow inside a cylinder whose wall undergoes a harmonic oscillation, and have successfully validated the quasi-steady solution experimentally for a cylinder with a dimensionless radius of $\eta_R = 35.7$. Similar to the solution for a flat plate or the flow outside of a cylinder, we show that the amplitude of fluid velocity oscillation decreases rapidly as we move away from the wall. This decay in amplitude is accompanied by a phase lag in velocity compared to the wall boundary as distance from the wall increases. At the
same time, the transient effects from the initial startup condition decay rapidly and almost vanish within the first cycle of oscillation.

Unique to our solution is the need to define two finite boundary conditions, the velocity at the outer wall and at the center of the cylinder. These boundaries result in a finite radial range and mean that the cylinder size influences the overall solution. The dimensionless radius of the cylinder, which is shown to be proportional to the product of Reynolds number and Strouhal number characterizing this flow, determines the thickness of the high velocity regions near the cylinder wall. For large cylinders, the penetration depth approaches Stokes’ solution for a flat plate. If the size of the cylinder approaches that of, or is smaller than, the characteristic viscous lengths scale of the flow, viscous forces dominate and solid body rotation within the cylinder results.

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Appendix A: Derivation with a sine wall boundary condition

Here we are demonstrating the derivation of the oscillatory flow inside a cylinder whose wall experiences an oscillation in the form of a sine function. Instead of Eq. (2a), we have a different boundary condition at the wall,

\[ u_\theta = a \cdot \sin(\omega t) \text{ at } r = R, \ t \geq 0. \]  \hspace{1cm} (A1)

Using the same nondimensionalization process, and then applying the Laplace transform, this boundary condition is written as

\[ F(\eta, s) = \frac{1}{s^2 + 1} \text{ at } \eta = \eta_R. \]  \hspace{1cm} (A2)

Solving the modified Bessel function in Eq. (7), with boundary conditions shown in Eq. (8b) and Eq. (A2), the dimensionless velocity after an inverse Laplace transform now becomes,
\[ f(\eta, \tau) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{1}{s^2 + 1} \cdot \frac{I_1(\eta \sqrt{s})}{I_1(\eta_R \sqrt{s})} \cdot \exp[s \tau] ds. \]  

(A3)

We then apply the Residue Theorem and sum up all non-zero residues to obtain a similar analytical solution. The solution for the dimensionless velocity for this boundary condition also contains a quasi-steady and transient part and is written as

\[
\begin{align*}
    f(\eta, \tau) &= \frac{1}{2} \left( \exp[i(\tau - \frac{\pi}{2})] \frac{I_1(\eta \sqrt{i})}{I_1(\eta_R \sqrt{i})} + \exp[-i(\tau - \frac{\pi}{2})] \frac{I_1(\eta \sqrt{-i})}{I_1(\eta_R \sqrt{-i})} \right) \\
    &\quad + \sum_{n} \exp[z_{0,n} \tau] \frac{4^{1/2}z_{0,n}^{1/2}}{(z_{0,n}^2 + 1)\eta_R} \cdot \frac{I_1(\eta_R \sqrt{z_{0,n}})}{I_0(\eta_R \sqrt{z_{0,n}}) + I_2(\eta_R \sqrt{z_{0,n}})}.
\end{align*}
\]  

(A4)

Here, we observe that the quasi-steady part is the same as the analytical solution for the cosine boundary condition except for a phase offset of \( \pi/2 \). Additionally, the transient part has a different numerator due to the gentle starting condition provided by the sine oscillation instead of the sudden starting condition created by the cosine function.

REFERENCES

15M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, 1965).

\[ u_\theta(R, t) = a \cdot \cos(\omega t) \]
Fitted center

Camera view

Fitted circle

PIV particles adhered to wall
\[ f \]

\[ \eta \]

- \( \tau = \pi/100 \) (dotted line)
- \( \tau = \pi/10 \) (red dashed line)
- \( \tau = \pi/5 \) (orange dashed line)
- \( \tau = \pi/2 \) (light orange line)
- Quasi-steady velocity at \( \tau = \pi/2 \) (black line with dots)

The graph shows the relationship between \( f \) and \( \eta \) for different values of \( \tau \).
(a) (b)

![Graphs showing angular velocity vs. radius and angular velocity vs. time](image)

- **Analytical solution at** $t = 2.5$ s
- **Analytical solution at** $t = 8.8$ s
- **PIV results at** $t = 2.5$ s
- **PIV results at** $t = 8.8$ s

---

**Graph (a)**: Angular velocity ($u_\theta$) vs. radius ($r$) with markers and error bars for analytical and PIV data.

**Graph (b)**: Angular velocity ($u_\theta$) vs. time ($t$) with a single line representing the analytical solution and markers for PIV results.

Note: The image is a representation of a scientific graph depicting the angular velocity ($u_\theta$) as a function of radius ($r$) and time ($t$) with a focus on comparing analytical solutions and PIV results.
(a) Graph showing the function $f_s$ vs. $\eta/\eta_R$ for different values of $\eta_R$: $\eta_R = 50$ (green solid line), $\eta_R = 10$ (red dashed line), $\eta_R = 5$ (dotted line), and $\eta_R = 1$ (purple dotted-dashed line).

(b) Graph showing the function $\delta^*/\delta$ vs. $\eta_R$. The slope is indicated as 0.99.

Legend:
- Red line: Inside a cylinder
- Black dashed line: Over a flat plate