Fossil field decay due to nonlinear tides in massive binaries

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\textbf{ABSTRACT}

\textit{Context.} Surface magnetic fields have been detected in 5\% to 10\% of isolated massive stars, hosting outer radiative envelopes. They are often thought to have a fossil origin, i.e. inherited from the stellar formation phase. Yet, magnetic massive stars are scarcer in (close) short-period binaries, as reported by the BinaMiC collaboration.

\textit{Aims.} Different physical conditions in the molecular clouds giving birth to isolated stars and binaries are commonly invoked. In addition, we propose that the observed lower magnetic incidence in close binaries may be due to nonlinear tides. Indeed, close binaries are likely prone to the tidal instability, a fluid instability growing upon the equilibrium tidal flow via nonlinear effects. Yet, stratified effects have been largely overlooked hitherto.

\textit{Methods.} We investigate theoretically and numerically the tidal instability in rapidly rotating, stably stratified fluids permeated by magnetic fields. We use the short-wavelength stability method to propose a comprehensive (local) theory of the tidal instability at the linear onset, discussing damping effects. Then, we propose a mixing-length theory for the mixing generated by the instability in the nonlinear regime. We successfully assess our theoretical predictions against proof-of-concept direct numerical simulations. Finally, we confront our predictions with the observations of short-period double-lined spectroscopic binary systems.

\textit{Results.} Using new analytical results, cross-validated by a direct integration of the stability equations, we show that the tidal instability can be generated by nonlinear couplings of inertia-gravity waves with the equilibrium tidal flow in short-period massive binaries, even against the Joule diffusion. In the nonlinear regime, a fossil magnetic field can be dissipated by the turbulent magnetic diffusion induced by the saturated tidal flows.

\textit{Conclusions.} We predict that the turbulent Joule diffusion of fossil fields would occur in a few million years for several short-period massive binaries. Therefore, turbulent tidal flows could explain the observed dearth of some short-period magnetic binaries.

\textbf{Key words.} hydrodynamics – instabilities – waves – stars: magnetic field – stars:massive

1. Introduction

1.1. Massive stars and magnetic incidence

The magnetism of massive stars has sparked the interest of astronomers for a long time (Babcock 1958). More recently, large spectropolarimetric surveys of these stars have been undertaken, such as MiMeS (Wade et al. 2015; Grunhut et al. 2016) or BOB (Hubrig et al. 2014). They have detected surface magnetic fields in 5\% to 10\% of pre-main sequence and main-sequence massive stars (e.g. Alecian et al. 2017; Mathys 2017). In addition, a magnetic dichotomy has been evidenced between the strong magnetic incidence in close binaries and weak magnetic incidence in isolated massive stars (e.g. Auriere et al. 2007; Sikora et al. 2018) and the ultra-weak magnetism of Vega-like stars (Lignieres et al. 2009; Petit et al. 2010, 2011; Blazère et al. 2016). The origin of these fields is unclear. According to stellar evolution theory, massive stars host thick outer radiative envelopes, which are stably stratified in density. These envelopes are often supposed to be motionless in standard stellar models (e.g. Kippenhahn et al. 1990). This severely challenges the classical dynamo mechanism (Parker 1979), which requires internal turbulent motions (e.g. convection in low-mass stars). Some dynamo mechanisms have been proposed, e.g. relying on the convection of the innermost convective core (Brun et al. 2005; Featherstone et al. 2009) generating magnetic flux tubes rising the stellar surface (MacGregor & Cassinelli 2003; MacDonald & Mullan 2004), on differentially rotating flows (Spruit 1999, 2002; Braithwaite 2006; Jouve et al. 2015) or on baroclinic flows (Simitev & Busse 2017). However, the relevance of these mechanisms remains elusive and debated.

The most accepted assumption is that magnetic fields in massive stars have a fossil origin (Borra et al. 1982; Moss 2001), since they appear relatively stable over the observational period. The fields would be shaped in the stellar formation phase and survive into later stages of stellar evolution. The fossil theory is now well supported by the existence of magnetic configurations stable enough to survive over stellar lifetime (Braithwaite & Spruit 2004; Braithwaite & Nordlund 2006; Reisenegger 2009; Duez & Mathis 2010; Duez et al. 2010; Akgün et al. 2013). Hence, the fossil theory may provide a unifying explanation for the magnetism of intermediate-mass stars (Braithwaite & Spruit 2017). However, the fossil hypothesis still suffers from several weaknesses. In particular, we may naively expect all massive stars to exhibit surface magnetic fields. This is not consistent with the observations (e.g. Alecian et al. 2017; Mathys 2017). Moreover, the theory does not explain convincingly the observed magnetic bi-modality (e.g. Auriere et al. 2007). To solve these issues, different physical conditions in the star-forming regions are usually invoked (e.g. Commerçon et al. 2010, 2011).
An efficient way to assess this hypothesis is to survey close binaries. Although the formation of binaries is not well understood, we can reasonably assume that the two binary components were formed together, under similar physical conditions. Then, observing magnetic fields in the two components of a binary system would provide constraints to disentangle initial condition effects from other possible physical effects. The BinaMicS collaboration (Alecian et al. 2014b) surveyed short-period massive binaries, aiming at providing new constraints on the magnetic properties of massive stars. About 170 short-period double-lined spectroscopic binary systems on the main-sequence have been analysed (Alecian et al. 2019). They have typical orbital periods $T_{\text{orb}} \leq 20$ days and a separation distance between the two components $D \leq 1$ au.

A magnetic incidence of about 1.5% has been measured in the BinaMicS sample (Alecian et al. 2017). This is much lower to what is typically found in isolated hot stars (see above). Therefore, radiative stars in short-period binary systems are apparently much less frequently magnetic than in isolated systems. This confirms the general trend observed in other studies, dedicated to intermediate-mass A-type stars (e.g. Carrier et al. 2002; Mathys 2017), and extends it to hotter and more massive stars. Note also that magnetic fields have been mostly observed only in one of the two components of the close binaries (Alecian et al. 2017), with a notable exception in $\epsilon$ Lupi system (Shultz et al. 2015).

If initial conditions were solely responsible for the presence of a fossil field, then we would naively expect fossil fields in the two components of a magnetic binary. This is clearly not a general trend. Thus, these puzzling observations defy the theories which are commonly invoked. Notably, is it due to formation processes (Commerçon et al. 2011; Schneider et al. 2016), that exclude more magnetic fields in binaries than in single stars? Or is there any other mechanism in close binaries, responsible for relatively quick dissipation of magnetic fields?

1.2. Mixing in radiative (stratified) interiors

An alternative scenario is to invoke some kind of mixing in the radiative envelopes, that may dissipate the pervading fossil fields dynamically. Identifying mixing sources in radiative stars is a long standing issue (see the review of Zahn 2008), since mixing also affects the transports of chemical elements and of angular momentum. Shear-driven turbulence, induced by the (expected) differential rotation of radiative envelopes (e.g. Goldreich & Schubert 1967; Rieutord 2006), has been largely investigated (e.g. Zahn 1974; Mathis et al. 2004, 2018).

A more efficient mixing in short-period stellar binaries may be provided by tides. Indeed, short-period binaries are strongly deformed (e.g. Chandrasekhar 1969; Lai et al. 1993). Tides proceed in two steps. First, they generate a quasi-hydrostatic tidal bulge, known as the equilibrium tide velocity field (Zahn 1966; Remus et al. 2012), leading to angular momentum exchange between the orbital and spinning motions. Second, they induce dynamical tides (e.g. Zahn 1975; Rieutord & Valdettaro 2010), i.e. waves that can propagate within the radiative regions. Radiative envelopes support the propagation of many waves, which are continuously emitted by various mechanisms (e.g. Gastine & Dintrans 2008a,b; Mathis et al. 2014; Edelmann et al. 2019). Among them, internal gravity waves (Dintrans et al. 1999; Mirouh et al. 2016) do induce mixing processes in radiative regions (Schatzman 1993; Rogers & McElwaine 2017).

However, the aforementioned tidal effects are only linear processes. They are likely relevant for weak tides in the solar and extra-solar planets (Ogilvie 2009), but they may be inefficient to modify fossil fields on their own. Moreover, nonlinear effects can significantly modify the outcome of the tidal response, and thereby the influence of tides on fossil fields. Indeed, the equilibrium tide flow can be unstable against the tidal instability in stars (e.g. Rieutord 2004; Le Bars et al. 2010; Barker & Lithwick 2013a,b; Clausen & Tilgner 2014; Barker et al. 2016; Barker 2016; Vidal & Cébron 2017; Vidal et al. 2018). This fluid instability is astrophysical version of the generic elliptical instability, which affects all rotating fluids with elliptically deformed streamlines (Bayly 1986; Pierrehumbert 1986; Waleffe 1990; Gledzer & Ponomarev 1992; Le Dizès 2000). The underlying physical mechanism is nonlinear triadic resonances between two waves and the background elliptical velocity (Kerswell 2002). Hence, in stellar interiors, the origin of such tidal instability is a resonance between rotational waves and the underlying strain field responsible for the elliptic deformation, i.e. the equilibrium tide flow (Remus et al. 2012). The nonlinear saturation of the tidal instability can exhibit a wide variety of nonlinear states in homogeneous fluids, such as space-filling small-scale turbulence (Le Reun et al. 2017, 2018) or even global mixing (Grannan et al. 2016; Vidal et al. 2018). Interestingly, Clausen & Tilgner (2014) have investigated the influence of the compressibility on the stability limits of tidal instability in stars or planets. They showed that the fluid compressibility has almost no effect on the tidal instability onset.

Yet, the fate of the tidal instability in stratified fluid interiors is poorly known. On the one hand, theoretical studies have shown that an axial density stratification, i.e. aligned with the spin angular velocity, has stabilising effects (Miyazaki & Fukumoto 1991, 1992). Moreover, in the equatorial regions, a radial stratification can either increase or decrease the growth rate of the instability (Kerswell 1993a; Le Bars & Le Dizès 2006; Cébron et al. 2013). On the other hand, three-dimensional numerical simulations suggest that the tidal instability is largely unaffected in stratified interiors, for a wide range of stratification (Cébron et al. 2010; Vidal et al. 2018). Therefore, a consistent global picture of the tidal instability in stably stratified interiors is highly desirable. Indeed, this is a prerequisite to assess the astrophysical relevance of the tidal instability for the stellar mixing in close massive binaries.

1.3. Motivations

The present study has a twofold purpose. First, we aim to propose a predictive global theory of the tidal instability in idealised stratified interiors. Such a theory should accurately predict the onset of the instability, reconciling within a single framework previous theoretical analyses (Miyazaki & Fukumoto 1992; Miyazaki 1993; Kerswell 1993a; Le Bars & Le Dizès 2006) and numerical studies (Cébron et al. 2010; Vidal et al. 2018). Then, asymptotic predictions for the (nonlinear) tidal mixing, as found numerically in Vidal et al. (2018), must be obtained to carry out the astrophysical extrapolation. Second, we aim to propose a new physical scenario of turbulent Joule diffusion of fossil fields, compatible with the observed lower magnetic incidence in short-period massive binaries as analysed by the BinaMicS collaboration (Alecian et al. 2017; Alecian & et al. 2019).

The paper is organised as follows. In §2, we present the idealised model. In §3, we investigate the linear regime of the tidal instability in stratified interiors. In §4, we develop a mixing-length theory of the (turbulent) tidal mixing, which is confronted against proof-of-concept simulations. Then, we attempt to propose a novel scenario for close binaries in §5, which is applied to short-period binary systems analysed by the BinaMicS collab-
oration. Finally, we end the paper with a conclusion in §6 and outline some perspectives.

2. Formulation of the problem

2.1. Assumptions

The full astrophysical problem is rather complex. Hence, we consider an idealised model, for which numerical simulations can be conducted and confronted against theory. We describe here the main assumptions, as they will be used throughout the paper. Our model retains the essential features for the tidal instability, i.e. rotation, stratification, magnetic fields and a tidally deformed geometry.

We consider a primary self-gravitating body, filled with an electrically conducting and rotating fluid of mass $M_1$ and volume $V$. Radiative fluid envelopes are expected to undergo differential rotation (Goldreich & Schubert 1967), e.g. provided by the contraction occurring during the pre-main-sequence phase or baroclinic torques (Busse 1981, 1982; Rieutord 2006). However, differential rotation tends to be smoothed out by hydromagnetic effects (e.g. Moss 1992). In particular, differential rotation may sustain the magneto-rotational instability, ultimately leading a state of solid-body rotation (Artl et al. 2003; Rüdiger et al. 2013, 2015) on a few Alfvén timescales (Jouve et al. 2015). Consequently, we assume that the radiative envelope is uniformly rotating.

Then, the primary is orbited by a companion star of mass $M_2$. We investigate here only short-period, non-coalescing binaries. Due to the combined action of rotation and of the gravitational potential, the shape of each binary component depart from the spherical geometry. We do not seek here the mutual tidal interactions between the primary and the secondary. Indeed, at the leading order, the primary (or the secondary) is a triaxial ellipsoid in solid-body rotation (e.g. Chandrasekhar 1969; Lai et al. 1993), as obtained by modelling the other component by a point-mass companion. Therefore, for the sake of simplicity, we treat the secondary as a point mass for the orbital dynamics (e.g. Hut 1981, 1982).

The secondary rises an equilibrium tide (Zahn 1966; Remus et al. 2012) on the fluid primary, with a typical equatorial amplitude denoted $\beta_0$. An initially eccentric binary system, with non-synchronised rotating components, evolutes towards an orbital configuration characterised by a circular orbit and, ultimately, the system will be synchronised (Hut 1981, 1982). For weakly elliptic orbits, Nduka (1971) showed that the ellipsoidal distortion $\beta_0$ points toward the tidal companion at the leading order. Vidal & Cébron (2017) also showed that weak orbital eccentricities have little effects on the internal fluid dynamics of the primary (at the leading order in the eccentricity). Thus, we assume that the binary system is circularised (or weakly eccentric), with an equatorial bulge aligned with the orbital companion.

Then, we consider only the leading-order component of the tidal potential, associated with the asynchronous tides (e.g. Chandrasekhar 1969). The other tidal components, e.g. obliquity tides, are mainly responsible for additional fluid instabilities which are superimposed on the tidal instability (e.g. Kerswell 1993b). They can be neglected in a first attempt.

Within the fluid primary, diffusive effects are of second order for the tidal instability, in the absence of significant surface diffusive effects at a free boundary (Rieutord 1992; Rieutord & Zahn 1997). Hence, we assume that the fluid has a uniform kinematic viscosity $\nu$, a radiative (thermal) diffusivity $\kappa_T$ (Kippenhahn et al. 1990) and a magnetic diffusivity $\eta = 1/(\mu_0 \sigma)$, where $\sigma$ is the electrical conductivity and $\mu_0$ the magnetic permeability of free space. Finally, Clausen & Tilgner (2014) showed that the fluid compressibility has almost no effect on the tidal instability. Therefore, we model density variations departing from the isentropic profile within the Boussinesq approximation (Spiegel & Veronis 1960).

2.2. Governing equations

The radiative star is modelled as a tidally deformed, uniformly rotating and stably stratified fluid domain in the Boussinesq approximation. The fluid domain, of typical density $\rho_0$, is rotating at the angular velocity $\Omega_s$ in the inertial frame. The orbital configuration is illustrated in figure 1. The orbital angular velocity in the inertial frame is denoted $\Omega_{orb}$, with $\Omega_{orb} \neq \Omega_s$ for a non-synchronised orbit. In the central frame, in which the boundary shape is stationary, the outer boundary $\partial V$ of the fluid domain describes an ellipsoid (e.g. Chandrasekhar 1969; Lai et al. 1993). Its mathematical expression in Cartesian coordinates $(x, y, z)$ is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1,$$

where $(a, b, c)$ are the semi-axes. The equatorial ellipticity is defined by $\beta_0 = \left[ a^2 - b^2 \right]/\left( a^2 + b^2 \right)$.

In the following, we work in dimensionless variables. To do so, we choose a typical radius $R$ of the fluid domain as unit of length, $\Omega_s^{-1}$ as a unit of time, $\Omega_{orb}^{-1} R/(\alpha T g_0)$ as unit of the temperature with $g_0$ a typical value of the gravity field at the outer boundary and $\alpha_T$ the thermal expansion coefficient (at constant pressure). For the magnetic field, we choose $\Omega_{orb} R^2 \mu_0 M_2$ as typical unit.

We also introduce the dimensionless orbital frequency $\Omega_0 = \Omega_{orb}/\Omega_s$. The dimensionless variables are the velocity field $v$, the temperature field $T$, the magnetic field $B$ and the gravity field $g$. They are written without $^*$, to distinguish them from their dimensional counterparts $[v^*, T^*, B^*, g^*]$. The field variables, at the position $r$ and time $t$, are governed in the rotating central frame by momentum, energy and induction equations. They read

$$\frac{\partial v}{\partial t} = -(v \cdot \nabla)v - 2\Omega_0 \mathbf{1}_z \times v - \nabla(P + P_m) + E k \nabla^2 v - T g + (B \cdot \nabla)B,$$

where $E = \nabla^2 + \kappa_T \partial T/\partial T$, $P_m = \kappa_T T$, and $\mathbf{1}_z$ is the unit vector parallel to the polar axis.

$$\frac{\partial g}{\partial t} = (B \cdot \nabla)\times B - (B \cdot \nabla)g - \sigma \nabla^2 g$$

$$\frac{\partial T}{\partial t} = \frac{1}{\kappa_T} \nabla^2 T - \frac{\beta}{\kappa_T} (B \cdot \nabla)g,$$

$$\frac{\partial B}{\partial t} = (B \cdot \nabla)\times B,$$

where $\beta = \kappa_T/(\rho_0 H_0)$.\n
$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \mathbf{v}) + \rho_0 \Omega_s^{-2} \mathbf{1}_z \times (\mathbf{1}_z \times \mathbf{v}) - \nabla \cdot \mathbf{j},$$

where $\mathbf{j}$ is the current density and $\nabla \cdot \mathbf{j} = 0$ due to the Boussinesq approximation.
Table 1: Typical values of the dimensionless numbers. CZ: stellar convective zones, e.g. in the Sun (Charbonneau 2014). Note that $N_0 = 0$ in convective envelopes. RZ: (rapidly) rotating radiative zones (e.g. Rieutord 2006). The order of magnitude of the Lehmann number in RZ has been estimated from the typical values for the scarce short-period magnetic binaries given in table 4.

<table>
<thead>
<tr>
<th>Number</th>
<th>Symbol</th>
<th>CZ</th>
<th>RZ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ekman</td>
<td>$Ek$</td>
<td>$10^{-16}$</td>
<td>$10^{-18}$</td>
</tr>
<tr>
<td>Prandtl</td>
<td>$Pr$</td>
<td>$10^{-6}$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>Magnetic Prandtl</td>
<td>$Pm$</td>
<td>$10^{-6}$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>Magnetic Lehmann</td>
<td>$Le$</td>
<td>$10^{-10}$</td>
<td>$10^{-12}$</td>
</tr>
<tr>
<td>Brunt-Väisälä</td>
<td>$N_0/\Omega_0$</td>
<td>0</td>
<td>0 – 100</td>
</tr>
<tr>
<td>Lehmann</td>
<td>$Le$</td>
<td>$10^{-5}$</td>
<td>$\leq 10^{-4}$</td>
</tr>
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with $P$ the hydrostatic pressure (including centrifugal effects), $P_m = |B|^2/2$ the magnetic pressure, $Q$ a heat source term and $g = -\nabla \Phi_0$ the (imposed) gravity field in the Boussinesq approximation. In governing equations (2), we have introduced as dimensionless numbers the Ekman number $Ek = \nu/(\Omega t R^2)$, the Prandtl number $Pr = \nu/\kappa \eta$, the magnetic Prandtl number $Pm = \nu/\eta$ and the magnetic Ekman number $Em = Ek/Pm$. Typical values are given in table 1 for stellar interiors. The latter are characterised by weakly diffusive conditions (i.e. $Ek, Ek/Pr, Ek/Pm \ll 1$). This regime will greatly simplify the analysis of the tidal instability.

We do not solve directly the full equations (2). Indeed, a reference ellipsoidal state is always first established, on which the tidal instability grows upon and satura
tes nonlinearly. We expand the field variables as perturbations (not necessarily small) around a steady reference ellipsoidal basic state $[U_0, T_0]$ (detailed in section 2.3). Thus, the dimensionless nonlinear governing equations for the perturbations $[u, \Theta](r, t)$ and the magnetic field $B(r, t)$ are

$$\frac{\partial u}{\partial t} = -(v \cdot \nabla) u + \frac{Ek}{Pr} \nabla^2 T + Q,$$  \hspace{1cm} (2b)

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + Em \nabla^2 B,$$  \hspace{1cm} (2c)

$$\nabla \cdot u = \nabla \cdot B = 0,$$  \hspace{1cm} (2d)

where $U_0$ is the (dimensional) strength of the fossil field. For the proof-of-concept simulations introduced in §4, the equations will be supplemented by appropriate boundary conditions.

2.3. Reference ellipsoidal configuration

We consider a steady reference equilibrium state, for which isopycnals coincide with isopotentials for $\Phi_0$ (including centrifugal force, self-gravity and tides). This assumption is consistent with compressible models (Lai et al. 1993). Hence, we assume that the background temperature profile $T_0(r)$ and the gravity field $g$, solutions of equations (2a)-(2b), are in a barotropic state (for a well-chosen $Q$) such that $g \times (\nabla T_0) = 0$. We neglect the baroclinic part, which is known to enhance the growth rate of the tidal instability in the equatorial plane (Kerswell 1993a; Le Bars & Le Dizès 2006). In the nonlinear regime, a baroclinic state would certainly also enhance the tidal turbulence in stellar interiors. However, we focus here on the less favourable configuration for the growth of the tidal instability (i.e. barotropic stratification). This choice is also consistent with the assumed uniform rotation of the fluid. Indeed, baroclinic torques are known to sustain differential rotation (e.g. Busse 1981, 1982; Rieutord 2006). Moreover, considering a barotropic state is a relevant assumption when the isopycnals move sufficiently fast to keep track of the rotating tidal potential (Le Reun et al. 2018). This situation is expected when the stratification has a large enough amplitude compared to the differential rotation $\Omega - \Omega_{orb}$ between the spin and the orbit.

To characterise the strength of stratification, we introduce the dimensional (local) Brunt-Väisälä frequency $N$ in the reference state. In dimensional variables, the latter is defined by

$$N^2 = -\alpha T g^* \cdot \nabla T^*.$$  \hspace{1cm} (4)

The fluid ellipsoid is assumed to be entirely stably stratified in density, i.e. $N^2 > 0$. The exact profiles in stellar interiors depend on the stellar internal processes. However, we aim at comparing analytical and numerical computations, which cannot be done for arbitrary profiles. Thus, we assume that the dimensionless total gravitational potential is quadratic, i.e.

$$\Phi_0 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}.$$  \hspace{1cm} (5)

Then, we consider the (dimensionless) reference temperature in barotropic equilibrium $T_0 = (N_0^2/\Omega_0^2) \Phi_0$, with $N_0$ a typical value of the Brunt-Väisälä frequency at the outer boundary. For intermediate-mass stars with $M_1 = 3M_\odot$ (where $M_\odot$ is the solar mass), a typical value is $N_0 \sim 10^{-3} \Omega^{-1}$ (e.g. Rieutord 2006), and typical values of $\Omega_0^{-1}$ range between 1 and 100 days (Mathys 2017). This yields $0 \leq N_0/\Omega_0 \leq 100$ in radiative stars. Hence, a barotropic reference configuration is a reasonable starting assumption.

Then, the ellipsoid is initially permeated by an fossil magnetic field $B_0(r)$ (in dimensionless form). To measure its relative strength (compared to rotation), we introduce the (dimensionless) Lehmann number (Lehnert 1954)

$$Le = \frac{B_0^*}{\Omega_0 R \sqrt{P_{DM} \mu_0}}.$$  \hspace{1cm} (6)

where $B_0^*$ is the typical (dimensional) strength of the fossil field. The Lehmann number is the ratio of the Alfvén and rotational velocities. When $Le \ll 1$, the Coriolis force dominates over the Lorentz force in momentum equation (2a). The regime $Le \ll 1$ is encountered in many magnetic stars (table 1). In the Sun, a typical value is $Le \sim 10^{-5}$ (Charbonneau 2014). For the scarce magnetic binaries which have been observed, the median field strength is $B_0^* \sim 1$ kG (see also values in table 4). This yields as typical values $Le \leq 10^{-3} - 10^{-4}$. Hence, we focus on the regime $Le \ll 1$ in the following.

Finally, the orbital configuration drives the equilibrium tidal flow (e.g. Remus et al. 2012). For non-synchronised orbits ($\Omega_0 \neq \Omega$),
1), its leading-order flow components in the central frame are (e.g. Cébron et al. 2012b; Vidal & Cébron 2017)
\[ U_0(r) = (1 - \Omega_0) - (1 + \beta_0) y \hat{1}_x + (1 - \beta_0) x \hat{1}_y . \] (7)

This is an exact incompressible solution of the nonlinear hydrodynamic momentum equation (2a). Moreover, it satisfies the non-penetration \( U_0 \cdot \hat{1}_r = 0 \) at the boundary \( \partial \Omega V \), with \( \hat{1}_r \) the unit outward normal vector. Note that the basic flow (7) is not rigorously a solution in the presence of an arbitrary magnetic field. Yet, the large-scale poloidal and toroidal components of \( B_0(r) \) are unlikely to modify the equilibrium tide flow in the weak field regime \( \varepsilon \ll 1 \) as often assumed (e.g. Kerswell 1993a, 1994; Mizerski & Bajer 2011).

3. Onset of the tidal instability

In this section, we present the stability analysis of the tidal instability at the linear onset. First, we outline the general stability method in §3.1. In §3.2, we carry out an asymptotic analysis to get a physical insight of the instability mechanism. The latter mechanism is confronted and validated against the solutions of the full stability equations in §3.3, without making any prior assumption. Finally, we discuss the (laminar) magnetic diffusive effects in §3.4.

3.1. Short-wavelength perturbations

In the absence of any driving mechanism, a fossil field \( B_0 \) slowly decays on the Ohmic diffusive timescale \( (\Omega, E_k / |m|)^{-1} \). This time is larger than the typical lifetime of least massive stars on the main-sequence (e.g. Braithwaite & Spruit 2017). However, equations (3) support the propagation of several waves, which can strongly modify the dynamical evolution of radiative envelopes. These waves are continuously emitted and, in the presence of tides, they can be nonlinearly coupled with the equilibrium tide velocity field \( U_0 \) to yield the tidal instability. The tidal instability is intrinsically a local (i.e. small scale) instability (Kerswell 2002; Cébron et al. 2012b; Barker & Lithwick 2013a,b), but also exists in global models (e.g. Kerswell 1993a; Granan et al. 2016; Vidal et al. 2018). The global stability analysis is beyond the scope of the present study. However, in the diffusionless regime, three-dimensional global perturbations of small enough length scales are excited (e.g. Vidal & Cébron 2017), such that they are not affected by the boundary. Hence, we can advantageously investigate the growth of the tidal instability in stellar interiors by performing a local stability analysis. In appendix A, we have extended the general local stability theory to account for combined magnetic and buoyancy effects within the Boussinesq approximation.

Rotating radiative interiors, characterised by \( \varepsilon \ll 1 \) and \( N_0 / \Omega_0 \gg 1 \) (see table 1), support the propagation of many waves. We focus on the subsonic wave spectrum (low Mach number), made of MAC (Magneto-Archimedean-Coriolis) waves. Indeed, high-frequency sonic waves are not involved in the tidal (elliptical) instability (Le Duc 2001), though they may be coupled with the tides (e.g. in coalescing binary neutron stars, see Weinberg 2016). The properties of MAC waves have already been outlined elsewhere (e.g. Gubbins & Roberts 1987; Mathis & de Brye 2011; Sreenivasan & Narasimhan 2017). Note that they have global bounded counterparts, known as Magneto-Archimedean-Coriolis (MAC) modes. The global modes are discussed in appendix B. The wave spectrum is bounded from below by slow Magneto-Coriolis (MC) waves, sustained by Lorentz and Coriolis forces with an angular frequency \( |\omega| \propto \varepsilon^2 \) (e.g. Malkus 1967; Labbé et al. 2015). The spectrum is bounded from above by internal gravity waves (modified by rotation), with an angular frequency \( |\omega_k| \leq \eta_0 / \Omega_0 \) for strong stratifications (Friedlander & Siegmann 1982a). In-between, the spectrum exhibits Coriolis waves (Greenspan 1968; Backus & Rieutord 2017) and inertial-gravity (or gravito-inertial) waves (e.g. Dintrans et al. 1999; Mirokh et al. 2016).

In the weak field limit \( \varepsilon \ll 1 \), magnetic effects are negligible (at the leading order) on inertial (Schmitt 2010; Labbé et al. 2015) and gravito-inertial waves, as outlined in appendix B. Moreover, only nonlinear couplings of inertial and gravito-inertial waves can trigger the tidal instability with significant growth rates to overcome the leading-order diffusive effects (Kerswell 1993a, 1994), as we confirm in appendix C. This behaviour is also supported by local (Barker & Lithwick 2013a) and global dynamo numerical simulations in homogeneous (Cébron & Hollerbach 2014; Reddy et al. 2018) and stratified fluids (Vidal et al. 2018). They showed that even a dynamo magnetic field only barely modifies the hydrodynamic tidal flows. Therefore, we can consider only the hydrodynamic Boussinesq stability equations in relevant the regime \( \varepsilon \ll 1 \).

We seek three-dimensional local perturbations, solution of the linearised equations (3). To do so, we consider short-wavelength (WKB) perturbations (Lifschitz & Hameiri 1991; Friedlander & Vishik 1991). They are local (plane-wave) perturbations, barely sensitive to the ellipsoidal boundary \( \partial V \), advected along the fluid trajectories \( X(t) \) of \( U_0(r) \). Given the basic tidal flow (7), the Eulerian three-dimensional perturbations are expressed as
\[ (u, \Theta)(r, t) = \bar{(u, \Theta)}(r, t) \exp (ik(t) \cdot r), \quad |k(t)| = |k_0|, \] (8)
where \( k(t) \) is the local wave vector with the initial value \( k_0 \). The local stability equations are solved in Lagrangian formulation, yielding the following ordinary differential equations (in dimensionless form)
\[ \frac{DX}{Dt} = U_0(X), \quad X(0) = X_0, \] (9a)
\[ \frac{Dk}{Dt} = - (\nabla U_0)^T k, \quad k(0) = k_0, \] (9b)
\[ \frac{D\bar{u}}{Dt} = \left( \frac{2 k k^T}{|k|^2} - I \right) \nabla U_0 + 2 \left( \frac{k k^T}{|k|^2} - I \right) \Omega_0 \hat{1}_z \times \bar{u} - \tilde{\Theta} \left( I - \frac{k k^T}{|k|^2} \right) g, \] (9c)
\[ \frac{D\tilde{\Theta}}{Dt} = -(\bar{u} \cdot \nabla) \tilde{\Theta}_0, \] (9d)
with \( D/Dt \) the Lagrangian time derivative. The solenoidal condition \( \bar{u} \cdot k = 0 \) is satisfied as long as it holds at the initial time, i.e. \( \tilde{\Theta}(0) \cdot k_0 = 0 \) in the Lagrangian description. Equations (9) do depend on the fluid trajectories \( X(t) \), because the gravity field \( g \) is spatially varying. Diffusive effects will be reintroduced in §3.4.

Equations (9) are ordinary differential equations along the Lagrangian trajectories \( X(t) \). They are also independent of the magnitude of \( k_0 \) in the diffusionless limit. Hence, we follow Le Dizès (2000) by restricting the initial wave vector to the unit spherical surface
\[ k_0 = \sin(\theta_0) \cos(\phi_0) \hat{1}_x + \sin(\theta_0) \sin(\phi_0) \hat{1}_y + \cos(\theta_0) \hat{1}_z, \] (10)
where $\phi_0 \in [0, 2\pi]$ is the longitude and $\theta_0 \in [0, \pi]$ is the colatitude between the spin axis $\mathbf{1}_z$ and the wave vector $\mathbf{k}_0$. In practice, equations (9) are integrated from a range of wave vectors $\mathbf{k}_0$ and initial positions $X_0$ within the reference ellipsoidal domain. The basic state is unstable with respect to short-wavelength perturbations if

$$\lim_{t \to \infty} \left( |\tilde{u}(t, X_0, k_0)| + |\tilde{\Theta}(t, X_0, k_0)| \right) = \infty. \quad (11)$$

Then, we determine the maximum (diffusionless) growth rate $\sigma$ as the fastest growing solution for all initial conditions, i.e. the largest Lyapunov exponent. This yields a sufficient condition for instability.

### 3.2. Asymptotic analysis

The equilibrium tidal flow (7) admits analytical periodic fluid trajectories $X(t)$ and wave vectors $k(t)$, solution of equations (9a)-(9b). To get physical insight of the instability mechanism, we carry out an asymptotic analysis in the limit $\beta_0 \leq 1$. We expand all quantities $(X, k, \tilde{u}, \tilde{\Theta})$ in successive powers of $\beta_0$ (see details in Le Dizès 2000).

#### 3.2.1. Triadic (nonlinear) couplings

It has been recognised for a long time that the tidal instability is a parametric instability in homogeneous (e.g. Bayly 1986; Waleffe 1990) and stratified fluids (e.g. Miyazaki & Fukumoto 1992; Miyazaki 1993). It is due to triadic interactions between pairs of waves that are coupled with the underlying tidal flow (7). At the leading asymptotic order ($\beta_0 = 0$), a necessary condition for a parametric tidal instability in rotating fluids is given by the resonance condition in the central frame (Kerswell 2002; Vidal & Cébron 2017)

$$|\omega_i - \omega_j + \delta| = 2|1 - \Omega_0|, \quad (12)$$

where $[\omega_i, \omega_j]$ are the angular frequencies of two free waves and $\delta$ a small detuning parameter, allowing for imperfect resonances (Kerswell 1993a; Le Dizès 2000; Lacaze et al. 2004; Vidal & Cébron 2017). The latter are due either to diffusive or topographic effects ($\delta \to 0$ for diffusionless fluids and weakly deformed spheres $\beta_0 \ll 1$). Detuning effects are negligible in the astrophysical regime (almost diffusionless and with $\beta_0 \ll 1$). Note that the case of synchronised orbits, characterised by $\Omega_0 = 1$ (in average), is forbidden by condition (12). Synchronised orbits must be treated separately (see appendix D).

Among the aforementioned resonances, sub-harmonic resonances are characterised by $\omega_i = -\omega_j$. Then, resonance condition (12) reduces (in the diffusionless regime) to

$$|\omega_i| = |1 - \Omega_0|, \quad (13)$$

which is a necessary stability condition for a sub-harmonic instability. Sub-harmonic resonances have been found to be the most unstable in homogeneous fluids (Kerswell 1993a, 1994; Le Dizès 2000; Vidal & Cébron 2017), i.e. yielding the largest growth rates.

We are now in a position to survey the possible nonlinear couplings of the different types of waves which yield tidal instabilities. The waves can be combined in several ways to fulfil condition (12) in non-synchronised systems. For instance, from condition (13), the tidal instability traditionally exists in the orbital range $-1 \leq \Omega_0 \leq 3$ when it involves Coriolis waves (e.g. Craik 1989; Le Dizès 2000; Vidal & Cébron 2017). We investigate in depth the coupling of hydrodynamic waves, postponing the discussion of hydromagnetic waves (which are unimportant for the present problem) to appendix C.

#### 3.2.2. Hydrodynamic waves at the parametric resonance

The behaviour of the tidal instability is intrinsically associated with the properties of the waves involved in the triadic resonances. The general wave-like equation, introduced in appendix B, is a mixed hyperbolic-elliptic partial differential equation. In the general case, a wave-like hyperbolic domain coexists with an elliptic domain, in which the waves are evanescent. At the leading asymptotic order $\beta_0 = 0$, Friedlander & Siegmann (1982b) showed that the characteristic curve delimiting the two domains is

$$(4 - \omega_i^2)\left| \omega_i^2 - (N_0/\Omega_0)^2 \right|^2 = 0. \quad (14)$$

The hydrodynamic wave spectrum is divided in two main regions. On the one hand, we have inertial waves modified by the gravity, called inertial-gravity waves and denoted $\mathcal{H}$. They have hyperbolic turning surfaces given by equation (14). They are sub-divided in two families

$$\mathcal{H}_1 : (N_0/\Omega_0)^2 < \omega_i^2 < 4, \quad (15a)$$

$$\mathcal{H}_2 : 0 < \omega_i^2 < \min[4, (N_0/\Omega_0)^2]. \quad (15b)$$

On the other hand, we have gravity waves modified by the rotation, called gravito-inertial waves and denoted $\mathcal{E}$. They have ellipsoidal turning surfaces given by equation (14). They are also divided in two families, characterised by

$$\mathcal{E}_1 : 4 < \omega_i^2 < (N_0/\Omega_0)^2, \quad (16a)$$

$$\mathcal{E}_2 : \max[4, (N_0/\Omega_0)^2] < \omega_i^2 < 4 + (N_0/\Omega_0)^2. \quad (16b)$$

These properties are quite general, since equation (14) depends solely on the reference state. Therefore, both global modes (e.g.
Dintrans et al. 1999) and local waves propagating upon this reference configuration exhibit this distinction.

The different families of waves satisfying the sub-harmonic resonance condition (13) are illustrated in figure 2. This is the main result of the linear theory, as this provides a necessary (and sufficient, see later) condition for the existence of the tidal instability (in both global and local models). Two kinds of tidal instabilities can be obtained, depending on the value of key parameter \( \Omega_0 \). At the leading asymptotic order, we managed to obtain a general expression for the sub-harmonic resonance condition (13) in the local theory, which can be written as

\[
\cos^2(\theta_0) = \frac{\bar{\omega} + N_0^2 r_0^2 [(N_0^2 r_0^2 - \bar{\omega}) \cos^2 a_0 - \cos(2a_0)]}{\bar{\omega}^2 + N_0^2 r_0^2 [N_0^2 r_0^2 - 2 \bar{\omega} \cos(2a_0)]} + 2 \sqrt{\omega_1 [\bar{\omega}(1 - N_0^2 z_0^2) + N_0^2 r_0^2 - 1]} \]

(17)

with the background rotation \( \bar{\omega} = \omega_0(1/1 - \Omega_0) \), \( \bar{N}_0 = (N_0/\Omega_0)/(1 - \Omega_0^2) \), \( \bar{\omega} = 4(1 + \Omega_0^2) \), the initial position \( X_0 = (x_0, z_0) = r_0(\sin a_0, \cos a_0) \) where \( r_0 \) is the initial radius and \( \omega_1 = N_0^2 r_0^2 \cos^2 a_0 \). The associated wave-like domains and colatitude angles \( \theta_0 \) are shown in figures 3 and 4.

The classical allowable range of the instability in homogeneous fluids is \(-1 \leq \Omega_0 \leq 3\) (Craik 1989; Le Dizès 2000). Within this range, the sub-harmonic condition involves only \( H \) waves, as shown in figure 2. For a neutral stratification (\( N_0 = 0 \)), they are inertial waves \( H_1 \), which propagate in the whole fluid cavity (Friedlander & Siegmann 1982b). They have the colatitude angle at the sub-harmonic resonance (Le Dizès 2000)

\[ 2 \cos(\theta_0) = \frac{1}{1 + \Omega_0} = 1 - \Omega_0. \]

(18)

This remains valid in weakly stratified fluids (i.e. \( N_0/\Omega_0 \ll 1 \)). Indeed, \( H_1 \) waves are only slightly modified by buoyancy. They still propagate in the whole fluid domain, as shown in figure 3a. In addition, their colatitude angle \( \theta_0 \) is slightly increased with respect to formula (18) on the polar axis.

When \( N_0/\Omega_0 \gg 1 \), \( H_1 \) waves morph into \( H_2 \) waves, i.e. inertia-gravity waves. These waves are strongly modified by buoyancy. Their wave-like domain is confined between hyperboloids, as shown in figure 3b. Outside the hyperboloid volume, these waves at the sub-harmonic resonance are evanescent (in global models). The characteristic curve delimiting the wave-like and evanescent domains is hyperbolic and given by equation (14). Along the rotation axis, local waves at the sub-harmonic resonance do not propagate in the evanescent regions for vertical positions \( z_e \), satisfying

\[ |z_e| \geq \frac{|1 - \Omega_0|}{N_0/\Omega_0}. \]

(19)

This shows that an axial stratification has a stabilising effect.

This behaviour is responsible for an equatorial trapping of the waves in the other directions at the sub-harmonic resonance. Indeed, the hyperbolic wave-like domain, bounded by equation (14), converges towards the conical volume delimited by the asymptotic limit \( \cos(\theta_e) = |1 - \Omega_0|/2 \) (Friedlander & Siegmann 1982b), where \( \theta_e \) is the critical colatitude. This is exactly formula (18). Therefore, expression (18) also defines the position of the critical latitudes at which the waves at the sub-harmonic resonance have a group velocity orthogonal to the gravity field (here radial direction at the leading order in \( \beta_0 \)), i.e. a wave vector \( k \perp g \). Hence, these specific waves are insensitive to the stratification. We emphasise that the presence of stratification does not alter the position of the critical latitudes (Friedlander & Siegmann 1982b,a). When \( |1 - \Omega_0| \to 0 \), the waves at the sub-harmonic resonance are equatorially trapped according to formula (18).

The orbital range \( \Omega_0 \leq -1 \) and \( \Omega_0 \geq 3 \) is known as the forbidden zone. In this range, any tidal instability must involve gravito-inertial waves \( E \) for the sub-harmonic mechanism, whatever the strength of the stratification. Indeed, figure 2 clearly shows that the waves at the sub-harmonic resonance depend only on the value of the orbital frequency \( \Omega_0 \). When \( N_0/\Omega_0 \leq 1 \), the sub-harmonic condition is never satisfied within this orbital range. Hence, no tidal instability is triggered.

However, gravito-inertial waves \( E \) can be excited at the sub-harmonic resonance for stronger stratifications, typically \( N_0/\Omega_0 \gg 1 \) when \( |\Omega_0| \) increases. Their critical characteristic surfaces, given by equation (14), are ellipsoidal. On the one hand, \( E_1 \) gravito-inertial waves are trapped in a region that does not encompass the polar axis, as shown in figure 4a. The minimum distance between the spin axis and the wave-like domain in the equatorial region is given by (Friedlander & Siegmann 1982b)

\[ x_c = \frac{\sqrt{|1 - \Omega_0|^2 - 4}}{N_0/\Omega_0}. \]

(20)

Therefore, the thickness of the wave-like domain increases when the ratio \( N_0/\Omega_0 \) increases. On the other hand, \( E_2 \) waves at the sub-harmonic resonance are gravito-inertial waves, trapped in a region that excludes the central part of the fluid (figure 4b). Along the polar axis, these waves do not propagate when \( z \) is smaller than the critical value (19). The size of wave-like domain increases when the ratio \( N_0/\Omega_0 \) increases. In the limit \( N_0/\Omega_0 \to \infty \), these waves become almost pure internal gravity waves, propagating in the whole fluid domain at the sub-harmonic resonance. This situation has been investigated numerically in local models (Le Reun et al. 2018), by assuming \( \Omega_0 = 0 \).

3.2.3. Asymptotic growth rate in the equatorial plane

At the next asymptotic order in \( \beta_0 \) (e.g. Le Dizès 2000), we can obtain a concise explicit formula for the growth rate \( \sigma \) of the tidal instability, valid in the equatorial plane. First, we focus on the equatorial plane \( z_0 = 0 \). Dispersion relation (17) yields for \( a_0 = \pi/2 \) (after simplification)

\[ \sqrt{\bar{\omega} + N_0^2 x_0^2 \cos(\theta_0)} = \pm 1 \]

(21)

with \( x_0 \leq 1 \) the position of the initial trajectory \( X_0 \) in the equatorial plane. In the particular case \( \Omega_0 = 0 \), equation (21) recovers equation (4.6) of Le Bars & Le Dizès (2006).

Several configurations are possible, depending on the parameters. On the one hand, the LHS of equation (21) is purely imaginary when \(-N_0^2 x_0^2 > \bar{\omega} \), i.e. when the stratification is unstably stratified (with \( N_0^2 / \Omega_0^2 \leq 0 \)). Then, a centrifugal instability grows upon the reference configuration, with a maximum (dimensionless) growth rate (e.g. Le Bars & Le Dizès 2006)

\[ \sigma = \frac{|1 - \Omega_0|}{N_0/\Omega_0} \sqrt{N_0^2 x_0^2 - \bar{\omega}}. \]

(22)

On the other hand, the tidal instability is triggered when all terms in equation (21) are real. Hence, no sub-harmonic instability
is possible when $\Omega_0^2 x_0^2 < -3 - 4\Omega_0 (2 + \Omega_0)$. This defines the forbidden zone of the tidal instability in stably stratified fluids, at a given position $x_0$. For neutral fluids ($N_0 = 0$), we recover the classical allowable orbital range of the tidal instability $-1 \leq \Omega_0 < 3$. Outside this range, we find that waves can be involved in triadic resonances in stratified fluids. Thus, sub-harmonic tidal instabilities could be triggered in stratified fluids when $\Omega_3 < -1$ and $\Omega_0 \geq 3$ (range known as the forbidden zone in neutral fluids). Then, the dimensionless growth rate in the equatorial plane is

$$\frac{\sigma}{|1 - \Omega_0|} = \frac{(2\Omega_0 + 3)^2}{16(1 + \Omega_0)^2 + 4N_0^2 x_0^2} \beta_0.$$  

(23)

Hence, the growth rate $\sigma$ is weakened by the stratification when $\overline{N}_0 x_0^2$ increases. This effect was already discussed in the conclusion of Le Bars & Le Dizès (2006), where they found that elliptical equipotentials are stabilizing contrary to circular equipotentials. However, in this former case, their equation slightly differs from equation (23). Actually, their formula is slightly erroneous as we confirm the validity of our equation (23) by direct numerical integration of the local stability equations (see below). Note also that equation (23) does not recover equation (24) of Cébron et al. (2013), obtained in the limit of a buoyancy force of order $\beta_0$. In this limit, we recover their erroneous formula (24) if we use their value for $\theta_0$, artificially set to its hydrodynamic value $\omega \cos^2 \theta_0 = 1$ instead of its correct value given by equation (21).

We show in figure 5 the maximum growth rate, computed from formula (23), by considering different orbital configurations $\Omega_0$. Several points are worthy of comment. First, tidal instability is excited in the equatorial regime when $-1 \leq \Omega_0 \leq 3$ (in the diffusionless limit), i.e. in the classical orbital range of the tidal instability (Le Dizès 2000). This mechanism occurs for any realistic value of $N_0/\Omega_0 \leq 100$ (see table 1). In this orbital range, the maximum growth rate is always obtained for neutral fluids ($N_0 = 0$), yielding the usual (dimensionless) growth rate (Le Dizès 2000)

$$\frac{\sigma}{|1 - \Omega_0|} = \frac{(2\Omega_0 + 3)^2}{16(1 + \Omega_0)^2} \beta_0.$$  

(24)

Second, outside the classical orbital range (i.e. in the forbidden zone), we unravel new tidal instabilities, triggered for large
when that a baroclinic state (i.e. enough values of the Brunt-Väisälä frequency (i.e. white line), the basic state is synchronised (see appendix D).

Formula (23) becomes by assuming an imposed gravity field with a different equatorial than reported in local stratified simulations (Le Reun et al. 2018).

The above condition (26) shows that the forbidden zone of the tidal instability coincides with the one for neutral fluid, i.e. \( \Omega_0 \leq -1 \) and \( \Omega_0 \geq 3 \). Outside this range, the asymptotic (dimensionless) growth rate is

\[
\sigma = \frac{(2\tilde{\Omega}_0 + 3)^2}{16(1 + \tilde{\Omega}_0)^2 - 4\tilde{N}_0^2 \tilde{z}_0^2} \beta_0. 
\]

Formula (27) is identical to the diffusionless growth rate devised by Miyazaki (1993), denoting \( \tilde{N}_0 \tilde{z}_0 \) their local value of the stratification. Hence, an axial stratification is uniformly stabilising along the polar axis.

3.3. Numerical solutions in the orbital range \( -1 \leq \Omega_0 \leq 3 \)

The previous asymptotic analysis shows that a stable stratification (\( N_0/\Omega_0 \geq 0 \)) has indubitably a stabilising behaviour in equatorial regions. Moreover, the axial stratification has a stabilising effect, i.e. is responsible for a trapping of the instability in equatorial regions. These observations are in agreement with existing local analyses (Miyazaki & Fukumoto 1992; Miyazaki 1993; Kerswell 1993a; Le Bars & Le Dizès 2006; Cébron et al. 2012b). However, this is barely consistent with three-dimensional numerical simulations (Vidal et al. 2018), showing that the growth rate at the onset is largely unaffected by the stratification. To reconcile these approaches, we investigate the onset of the tidal instability in the whole reference fluid domain.

To go beyond the analytical formulas in the equatorial plane and on the polar axis, we solve numerically local stability equations (9). To do so, we have used the local stability code SWAN (Vidal & Cébron 2017). We have updated it to handle the general local stability equations, which are described in appendix A. Moreover, by solving numerically the full local equations, we do not assume a priori sub-harmonic condition (13). Hence, we emphasise that the numerical solutions will assess the general validity of sub-harmonic condition (13) in stratified fluids, which has already been confirmed in homogeneous fluids (Kerswell 1993a, 1994; Le Dizès 2000; Vidal & Cébron 2017).

In the astrophysical regime \( \beta_0 \ll 1 \), the resonance condition (12) or (13) (if valid), are satisfied numerically for only a few initial wave vectors \( k_0 \). Numerically, this is too expansive to survey all the possible configurations for \( k_0 \). Thus, we set the equatorial ellipticity to the value \( \beta_0 = 0.2 \). This does not change in any way the relevance of the following numerical results, because \( \sigma \) is proportional to \( \beta_0 \) (when \( \beta_0 \ll 1 \)). However, for large values of \( \beta_0 \), the general resonance condition (12) can be satisfied for a wider range of initial wave vectors \( k_0 \) due to geometrical detuning effects (Le Dizès 2000; Vidal & Cébron 2017). Hence, the computations are more tractable numerically.

In practice, we have considered a large enough number of fluid trajectories \( X(t) \) and \( k_0 \), sampling the whole ellipsoidal domain to get representative results.

We have validated the code against analytical formulas (23) and (27), obtaining a perfect agreement and cross-validating the asymptotic analysis (not shown). Then, we only investigate the stability of the equilibrium tide flow (7) within the orbital range \(-1 \leq \Omega_0 \leq 3 \), in which the binary systems considered in §5 belong. When the stratification is neutral (\( N_0 = 0 \)), the whole domain is unstable as expected (not shown), with a homogeneous growth rate predicted by formula (24). We survey illustrative stably stratified configurations \( N_0/\Omega_0 \geq 0 \) in figure 6. Several aspects of the figure are worthy of comment. We clearly recover the trapping of the instability by the axial stratification, outlined
$N_0/\Omega_0 = 1$ (H₁)  

$N_0/\Omega_0 = 2$ (H₂)  

$N_0/\Omega_0 = 5$ (H₂)  

(a) $\Omega_0 = -0.5$

$N_0/\Omega_0 = 1$ (H₁)  

$N_0/\Omega_0 = 2$ (H₂)  

$N_0/\Omega_0 = 10$ (H₂)  

(b) $\Omega_0 = 0$

$N_0/\Omega_0 = 1$ (H₂)  

$N_0/\Omega_0 = 2$ (H₂)  

$N_0/\Omega_0 = 5$ (H₂)  

(c) $\Omega_0 = 0.5$

Fig. 6: Largest normalised growth rate $\sigma/\beta_0$ for several configurations, computed with the SWAN code. Ellipsoidal boundary of ellipticity $\beta_0 = 0.2$. Visualisations in a meridional plane with normalised axes $x/a$ and $z/c$, with $a = \sqrt{1 + \beta_0}$, $b = \sqrt{1 - \beta_0}$ and $c = 1/(ab)$. White dashed lines predicted by formula (18) show the critical latitudes, on which the growth rate is maximum and given by (24). For each case, the type of waves involved in parametric mechanism is specified between brackets. Dashed (gray) curves delimitate the domain of existence of H₂ waves at the resonance (in the regime $\beta_0 \ll 1$).
by the weakening of the growth rate in formula (27). In the bulk, the first weakening occurs near the polar regions, and then spreads out towards lower latitudes when \(N_0/\Omega_2\) increases (from top to bottom panels in figure 6). Along the polar axis, it turns out that the transition between unstable and stable areas occurs at position (19). In addition, the equatorial region is still unstable for the range of \(N_0/\Omega_2\) considered, in agreement with figure 5. Then, the numerical analysis unravels an unexpected feature compared to the asymptotic analysis. When \(N_0/\Omega_2\) increases, tidal instabilities are always triggered in the bulk. Non-vanishing growth rates exist as long as waves can be nonlinearly coupled, according to the resonance condition valid when \(\beta_0 \ll 1\) (bounded from below and above by the gray dashed curves). An exception appears here for \(\Omega_0 = -0.5\) and \(N_0/\Omega_2 = 5\) in figure 6a. This is due the finite value \(\beta_0 = 0.2\) used in the numerics, which is responsible for imperfect resonances in condition (12) due to geometric detuning effects (e.g. Le Dizès 2000; Lacaze et al. 2004; Vidal & Cébron 2017). Moreover, the striking feature is that stratification tends to confine the tidal instability along critical (conical) latitudes (white dashed lines), tilted from the spin (polar) axis. The tilt angle in the numerics is exactly the colatitude angle \(\theta_0\) (given our numerical resolution, not shown), predicted by formula (18) and which maximizes the classical tidal instability for neutral fluids (\(N_0 = 0\)). This shows that the equatorial trapping does not affect similarly all the orbits. When \(-1 \leq \Omega_0 \leq 1\), the tilt angle \(\theta_0\) given by formula (18) goes from \(\theta_0 = 0\) to \(\theta_0 = \pi/2\). Hence, the instability on retrograde orbits (with small values of \(\theta_0\)) is less weakened than on prograde orbits. When \(N_0/\Omega_2 \gg 1\), the tidal instability is equatorially trapped between the conical layers, with growth rates in the equatorial plane predicted by formula (23). However, on these conical layers, it turns out that the largest growth rate \(\sigma\) is unaffected by the stratification, for any value of \(N_0/\Omega_2\). Hence, the maximum growth rate of the tidal instability in stratified fluids is always given by formula (24), for any orbit in the orbital range \(-1 \leq \Omega_0 \leq 3\).

Therefore, the numerical analysis has confirmed and extended the asymptotic analysis. In stably stratified interiors, resonance condition (13) illustrated in figure 2 is a necessary and sufficient condition for the tidal instability (when \(\beta_0 \ll 1\)). Indeed, we have not found any other resonance yielding larger growth rates compared to the sub-harmonic resonances. In the orbital range \(-1 \leq \Omega_0 \leq 3\), the tidal instability is triggered by sub-harmonic resonances of inertia-gravity waves. Moreover, there is an equatorial trapping of the tidal instability between conical latitudes, depending on the orbital configuration according to formula (18). At these latitudes, the wave vector is parallel to the gravity field, such that the maximum growth rate \(\sigma\) is unaffected by the stable stratification.

3.4. Leading-order (laminar) diffusive effects

We reintroduce now the leading-order (laminar) diffusive effect at the onset of the tidal instability. In the diffusive regime, the tidal instability is triggered if the largest diffusionless growth rate \(\sigma\) overcomes the (negative) laminar damping rates due to viscosity \(\tau_v\), radiative diffusivity \(\tau_r\) and Joule diffusion \(\tau_J\). Hence, the diffusionless growth rate \(\sigma\) ought to be reduced by the laminar damping rates, yielding the diffusive growth rate

\[
\sigma_D = \sigma - (\tau_v + \tau_r + \tau_J). \tag{28}
\]

We have confirmed in §3.3 that the tidal instability is a parametric instability, involving only inertial and/or gravito-inertial waves in radiative interiors. Consequently, we can simply estimate the laminar damping rates by computing the damping rates of the inertial and gravito-inertial waves involved in triadic couplings. Indeed, triadic couplings can only yield non-vanishing growth rates (28) if the waves individually exist, i.e. if they are not damped by any diffusive effect before being efficiently nonlinearly coupled. We have shown that in §3.3 that the diffusionless growth rate \(\sigma\) is maximum on critical latitudes, where the wave vector satisfies \(k_0 \times g = 0\) (when \(\beta_0 \ll 1\)). Then, in the local plane-wave model, the buoyancy term in the local vorticity equation (which is proportional to \(k_0 \times g\)) vanishes such that vorticity and energy equations are then uncoupled (in the local formalism). This means that these waves are locally insensitive to the stratification on the critical latitudes, yielding \(\tau_v = 0\). Thus, in the absence of background turbulent motions (see the discussion in §3.5), the waves are individually damped by viscosity and Joule diffusion (in the weak field regime \(Le \ll 1\)).

For the stability computations, we rewrite here the magnetic field as

\[
B = B_0 + b, \tag{29}
\]

where the fossil field \(B_0\) is assumed to be steady here. The pervading fossil magnetic fields are nearly axisymmetric and dipole-dominated at the leading order, as observed in magnetic binaries (e.g. Alecian et al. 2016; Landstreet et al. 2017; Kochukhov et al. 2018; Shultz et al. 2017, 2018). For the stability computations, we assume a fossil field of the form \(B_0 \propto 1\), with a dimensionless strength measured by the Lehnert number \(Le\). The presence of other field components only slightly modifies the frequencies of the inertial and inertial-gravity waves at the onset. There is no reason to suggest that the damping rate would behave qualitatively otherwise in the laminar regime. In the weak field regime \(Le \ll 1\), the damping rates have been devised by Sreenivasan & Narasimhan (2017) in the local theory and by Kerswell (1994) in the global one. They depend on the wave properties, i.e. here the wave vector. Notably, we explain in appendix C why the mixed couplings between inertial waves and slow MC waves cannot lead to any tidal instability in short-period binaries (in the presence of Joule diffusion). Hence, we remind the reader that only parametric resonances of inertial and gravito-inertial waves can generate the tidal instability in the presence of magnetic fields.

Then, the viscous and the Joule damping rates in the weak field regime \((Le \ll 1)\) in any \(z\)-plane read

\[
\begin{align*}
\tau_v &= -|k_0|^2 \frac{E_k}{\bar{E}_m} (30a) \\
\tau_J &= -\frac{4 \cos^2(\theta_0)}{\cos^2(\theta_0)+|k_0|^2} \left(\frac{E_m}{\bar{E}_m}\right)^2 |1-\Omega_0|, \tag{30b}
\end{align*}
\]

with \(|k_0|\) the norm of the wave vector at the resonance (and at initial time) and \(\cos(\theta_0)\) given by condition (18). Note that expression \(30b)\) is quantitatively valid when \(B_0 \propto 1\). (Sreenivasan & Narasimhan 2017). In the regime \(Pm \ll 1\), laminar Joule diffusion is the leading-order dissipative effect (\(|\tau_J| \gg |\tau_v|\)). Joule damping has already been considered for homogeneous fluids (Kerswell 1994; Herreman et al. 2009, 2010; Cébron et al. 2012b). Note that formula \(30b)\) is exactly the Joule damping rate of the tidal instability in neutral fluids (\(N_0 = 0\)). Besides, formulas of Herreman et al. (2009) and Cébron et al. (2012b) are recovered in the limit \(|k_0| \gg 1\) by using the resonance condition to set \(\theta_0\), i.e. \(2 \cos \theta_0 = \pm 1\) for \(N_0 = 0\). Formula \(30b)\) has two asymptotic behaviours, depending on the value of \(k_0\). They are separated by the condition

\[
|k_0| = \sqrt{2\cos(\theta_0)/E_m} \sim E_m^{-1/2}. \tag{31}
\]
On the one hand, we obtain a wave-dominated regime when $|k_0| \lesssim Em^{-1/2}$, in which the Joule damping rate scales as $\tau_\Omega \propto -Em Le^2 |k_0|^4/4$. On the other hand, we get a diffusion-dominated regime when $|k_0| \gtrsim Em^{-1/2}$. In the latter regime, the damping rate is independent of the wave vector and scales as $\tau_\Omega \propto -Le^2/Em$.

We illustrate in figure 7 the evolution of the Joule damping rate (30b) in the different regimes. The tidal instability will survive in the presence of magnetic fields if $\sigma \gg |\tau_\Omega|$. Typical values of the diffusionless growth rate, given by formula (24), are $\sigma \sim O(\beta_0)$ with $\beta_0 \in [10^{-4}, 10^{-2}]$ in close binaries. Computations at $Le = 10^{-5}$ and $Ek/Pm = 10^{-12}$ for a dimensionless fossil field $B_0 = I_z$ aligned with the spin axis.

3.5. Other dissipative mechanisms

At the linear onset, the laminar diffusive effects discussed in §3.4 are always present, but we have shown that they are (rather) small compared to the largest diffusionless growth rate $\sigma$. Hence, these effects can be reasonably neglected at the onset, yielding $\sigma_D \approx \sigma$. However, other diffusive effects do exist in stellar interiors, which may weaken the growth of the tidal instability.

Phase mixing is known to provide a significant source of Joule heating, by dissipating Alfvén (and magneto-sonic) waves e.g. in stellar atmospheres (e.g. Heyvaerts & Priest 1983) or stellar interiors (Spruit 1999). Yet, phase-mixing is likely irrelevant for the tidal instability in the weak field regime ($Le \ll 1$), notably because Alfvén waves are not involved in the tidal instability (see appendix C). Whether phase-mixing could enhance the dissipation of inertial and gravito-inertial waves in stellar interiors remains unknown and is largely beyond the scope of the present study.

In the presence of an innermost convective envelope, inertial and gravito-inertial waves can exhibit singular shear layers, reminiscent of wave attractors (e.g. Dintrans et al. 1999; Rieutord & Valdettaro 2010; Mirosh et al. 2016; Lin & Ogilvie 2017; Rieutord & Valdettaro 2018). These global wave patterns, which are not directly involved in the parametric mechanism of the tidal instability, fill the whole fluid domain and may provide an additional bulk damping rate for the tidal instability. Indeed, these structures can be destabilised in the nonlinear regime (Jouve & Ogilvie 2014), possibly yielding small-scale instabilities. Brunet et al. (2019) showed that the resulting small-scale turbulence in the bulk could be well modelled by a turbulent eddy diffusion. In particular, anisotropic shear-driven turbulence may be generated (e.g. Zahn 1992). In such a case, Garaud et al. (2017) and Gagnier & Garaud (2018) proposed to model the local shear-driven turbulence by introducing the turbulent viscosity

$$\nu_t \approx 0.08 \kappa T / J, \quad J = N_0^2 / S^2,$$

with $\kappa T$ the radiative diffusivity, $J$ the local gradient Richardson number and $S$ the local shearing rate (responsible for the shear instabilities). The stability criterion for shear instabilities is apparently $J Pr \lesssim 0.007$ (Garaud et al. 2017). Then, prediction (32) would yield an effective turbulent Ekman number $Ek_t \leq 10^{-10}$ for plausible stellar values, to use in expression (30a) for the viscous damping rate. For the range of wave numbers $|k_0|$ given in figure 7, we find that the associated turbulent damping rate is negligible compared to the diffusionless growth rate $\sigma$ (not shown). Therefore, even in the presence of shear-driven instabilities, the associated turbulent damping can be safely ignored at the onset of the tidal instability for the (strong enough) tidal deformations considered in this work ($\beta_0 \sim 10^{-3} - 10^{-2}$, see table 2).

4. Turbulent mixing due to nonlinear tidal flows

At this stage, we have shown that the tidal instability can be triggered within stably stratified interiors, even against the stabilising effect of a background (fossil) magnetic field in the weak field regime $Le \ll 1$. The next step is to characterise the saturated regime of the tidal flows. Modelling turbulent mixing in radiative interiors is one of the enduring problems in stellar dynamics (e.g. Zahn 1974). Several studies have examined the turbulence in radiative zones (e.g. Zahn 1992; Mathis et al. 2004; Garaud et al. 2017; Gagnier & Garaud 2018; Mathis et al. 2018). Yet, these models focus on shear-driven turbulence. Hence, the tidally driven turbulence in binaries remains to be described. Numerical simulations have shown that small-scale turbulence can be excited by the tidal instability (Barker & Lithwick 2013a,b; Le Reun et al. 2017), possibly leading to global tidal mixing (Vidal et al. 2018). Thus, tidal mixing is expected in radiative interiors. We motivate our assumptions in §4.1. Then, we use dimensional-type arguments in §4.2 to develop a phenomenological description of the nonlinear tidal mixing in radiative interiors in §4.3, valid in the orbital range $-1 \leq \Omega_0 \leq 3$. Finally, we assess its validity by using proof-of-concept simulations in §4.4.
$B_0$ is not much affected by the tidal flow, which is not expected to generate significant mixing. It only decays on the slow (laminar) Joule diffusion time, which is much larger than the time scale of for the onset of the tidal instability for stellar parameters. This phenomenon is well-known in global models of resistive magnetohydrodynamics, also known as free-decay of magnetic fields (e.g. Moffatt 1978). However, in the saturated regime, the fossil field would interact nonlinearly with the nonlinear tidal flows, as governed by induction equation

$$\frac{\partial B}{\partial t} = \nabla \times [(U_0 + u) \times B] + \frac{E_k}{Pm} \nabla^2 B,$$

(33a)

$$\nabla \cdot B = 0, \quad B(r, t = 0) = B_0(r),$$

(33b)

in which the initial time $t = 0$ refers now to an initial time just after the growth of the instability. In equation (33a), the nonlinear velocity field $u$ is governed by momentum equation (3a). In the relevant weak field regime $Le \ll 1$, nonlinear numerical simulations of the coupled problems showed that magnetic effects do not weaken the turbulent tidal flows (Barker & Lithwick 2013b; Cébron & Hollerbach 2014; Vidal et al. 2018). These turbulent flows generate mixing, that would ultimately enhance the Ohmic diffusion of the fossil field $B_0$. Therefore, Ohmic diffusion ought to be increased (a priori), which is often modelled by introducing a turbulent magnetic diffusivity (e.g. Kitchatinov et al. 1994; Yousef et al. 2003; Käpylä et al. 2019). In this configuration, the initial fossil field is expected to decay on somehow faster time scales, due to the presence of the nonlinear mixing generated by the tidal instability. This situation strongly differs from the picture of ideal magnetohydrodynamics, in which the laminar decay of the fossil field is small (and so can be sometimes neglected).

Note that an initial fossil field may still be in quasi-equilibrium with the tidal fields, if the dissipated field is continuously regenerated by some kind of dynamo action. However, dynamo action of tidal flows in strongly stratified interiors remains elusive (Vidal et al. 2018) and will not be investigated here. Consequently, to estimate the fossil field decay due to the tidal instability, we must estimate the turbulent magnetic diffusivity generated by the nonlinear tidal flows (Vidal et al. 2018). Thus, we may replace any laminar diffusivity (denoted $\mathcal{D}$) by an effective eddy diffusivity (denoted $\mathcal{D}_f$), induced by the nonlinear tidal flows. Then, a mixing-length theory (e.g. Tennekes & Lumley 1972) yields in dimensional form (up to a unknown proportional constant)

$$\mathcal{D}_f \propto \frac{1}{3} u_t l_t,$$

(34)

where $u_t$ and $l_t$ are respectively the typical (dimensional) local velocity and length scale of the turbulent motions. Note that $u_t$ is the typical amplitude of the nonlinear tidal flows. This must not be confused with the amplitude $u_0$ of the waves (e.g. excited by the tidal forcing in the linear phase). Indeed, $u_0$ is much smaller than $u_t$ in amplitude (e.g. Rogers & McElwaine 2017). Hence, the eddy diffusivity $\mathcal{D}_f$ is a local property of the nonlinear flow, rather than directly a property of the fluid or the tidal forcing. The key point to apply formula (34) is to find accurate predictions for $u_t$ and $l_t$ for nonlinear tidal flows.

On the one hand, we have shown in §4.3 that the tidal instability is generated by sub-harmonic resonances of inertial waves, more or less modified by the gravity field in the orbital range $-1 \leq \Omega_0 \leq 3$. This mechanism holds whatever the local strength of the stratification, measured by the ratio $N_0/\Omega_s$. This suggests that the turbulent velocity scale $u_t$ should not depend (strongly) on the strength of the local stratification (4), i.e. the typical ratio $N_0/\Omega_s$. This is supported by proof-of-concept simulations (see figure 2b of Vidal et al. 2018), showing that nonlinear tidal flows exhibit the scaling devised in homogeneous fluids (Barker & Lithwick 2013a; Granman et al. 2016), i.e.

$$u_t \sim \alpha_1 \beta_0 r_1 \Omega_s (1 - \Omega_0),$$

(35)

with $\eta \leq R$ the local position and $\alpha_1 \sim 0.3 - 0.5$ a dimensionless pre-factor obtained numerically both in homogeneous (Granman et al. 2016, estimated from figure 4d) and strongly stratified tidal flows (Vidal et al. 2018, estimated from figure 2b). Hence, we can safely estimate the turbulent velocity $u_t$ by using prescription (35). On the other hand, $l_t$ should depend on the local ratio $N_0/\Omega_s$. Several regimes have been found in forced stratified turbulence (e.g. Brethouwer et al. 2007).

4.2. Mixing-length theory

Estimating a realistic turbulent magnetic diffusivity is hampered by the fact that no numerical model cannot probe accurately the stellar conditions. This makes the relevance of numerical results sometimes elusive. Therefore, we aim to build asymptotic scaling laws of the tidal mixing, based on dimensional-type arguments that embrace both numerical and stellar conditions. To predict the local tidal mixing in stratified interiors, we attempt to develop a mixing-length theory, by analogy with mixing-length arguments commonly used for shear-driven turbulence in radiative interiors of stars (e.g. Zahn 1992; Mathis et al. 2004, 2018).

In turbulent flows, the laminar viscosity is often replaced by an effective eddy (turbulent) viscosity, usually modelled by using mixing-length theories in stellar contexts. In hydromagnetic turbulence, Yousef et al. (2003) and Käpylä et al. (2019) argued that in the weak field regime ($Le \ll 1$) the turbulent magnetic Prandtl number is not far from unity. This suggests that the turbulent magnetic diffusivity can be also modelled by mixing-length type predictions. This is supported by local hydromagnetic simulations of the three-dimensional turbulence generated by the tidal instability (Barker & Lithwick 2013b), showing that weak magnetic fields can even favour the small-scale tidal stratified models. Global tidal mixing has also been found global stratified models (Vidal et al. 2018). Thus, we may replace any laminar diffusivity (denoted $\mathcal{D}$) by an effective eddy diffusivity (denoted $\mathcal{D}_f$), induced by the nonlinear tidal flows. Then, a mixing-length theory (e.g. Tennekes & Lumley 1972) yields in dimensional form (up to a unknown proportional constant)

$$\mathcal{D}_f \propto \frac{1}{3} u_t l_t,$$

(34)

where $u_t$ and $l_t$ are respectively the typical (dimensional) local velocity and length scale of the turbulent motions. Note that $u_t$ is the typical amplitude of the nonlinear tidal flows. This must not be confused with the amplitude $u_0$ of the waves (e.g. excited by the tidal forcing in the linear phase). Indeed, $u_0$ is much smaller than $u_t$ in amplitude (e.g. Rogers & McElwaine 2017). Hence, the eddy diffusivity $\mathcal{D}_f$ is a local property of the nonlinear flow, rather than directly a property of the fluid or the tidal forcing. The key point to apply formula (34) is to find accurate predictions for $u_t$ and $l_t$ for nonlinear tidal flows.

On the one hand, we have shown in §4.3 that the tidal instability is generated by sub-harmonic resonances of inertial waves, more or less modified by the gravity field in the orbital range $-1 \leq \Omega_0 \leq 3$. This mechanism holds whatever the local strength of the stratification, measured by the ratio $N_0/\Omega_s$. This suggests that the turbulent velocity scale $u_t$ should not depend (strongly) on the strength of the local stratification (4), i.e. the typical ratio $N_0/\Omega_s$. This is supported by proof-of-concept simulations (see figure 2b of Vidal et al. 2018), showing that nonlinear tidal flows exhibit the scaling devised in homogeneous fluids (Barker & Lithwick 2013a; Granman et al. 2016), i.e.

$$u_t \sim \alpha_1 \beta_0 r_1 \Omega_s (1 - \Omega_0),$$

(35)

with $\eta \leq R$ the local position and $\alpha_1 \sim 0.3 - 0.5$ a dimensionless pre-factor obtained numerically both in homogeneous (Granman et al. 2016, estimated from figure 4d) and strongly stratified tidal flows (Vidal et al. 2018, estimated from figure 2b). Hence, we can safely estimate the turbulent velocity $u_t$ by using prescription (35). On the other hand, $l_t$ should depend on the local ratio $N_0/\Omega_s$. Several regimes have been found in forced stratified turbulence (e.g. Brethouwer et al. 2007).

4.3. Phenomenological prescriptions

4.3.1. Weakly stratified regime ($N_0/\Omega_s \leq 1$)

In the weakly stratified regime, characterised by $N_0/\Omega_s \leq 1$, ‘$\mathcal{H}_1$ waves at the sub-harmonic resonance are barely affected by the stratification. We estimate $l_t$ by balancing the nonlinear term $(u \cdot \nabla) u$ with the injection term $(u \cdot \nabla) U_0$ in the momentum equation (3a). This yields the typical turbulent length scale in dimensional form $l_t \propto \alpha_1 r_1$. Hence, the eddy diffusivity in the weakly stratified regime yields the diffusion coefficient (in dimensional form)

$$\mathcal{D}_f \propto \frac{1}{3} \alpha_1^2 \beta_0 r_1^2 \Omega_s (1 - \Omega_0).$$

(36)

Formula (36) predicts a roughly homogeneous mixing in the weakly stratified regime, as found in global models (Granman et al. 2016; Vidal et al. 2018) in which $\eta \approx R$. This explains why the tidal mixing computed in Vidal et al. (2018) is roughly constant as a function of the stratification when $N_0/\Omega_s \leq 1$ (see their figure 9). However, estimate (36) may be reduced in this due to (compressible) density variations (close to the isentropic profile when $N_0/\Omega_s \ll 1$).

Finally, formula (36) provides a good estimate of the leading-order term in the eddy diffusivity tensor (e.g.
Dubrulle & Frisch (1991; Wirth et al. 1995). In addition, note that rotation would likely support small anisotropic diffusion in the axial direction (Tilgner 2004; Elsner & Rüdiger 2007).

4.3.2. Stratified regimes \((N_0/\Omega_s \geq 1)\)

We now investigate the stratified regimes \(N_0/\Omega_s \geq 1\). Stratified turbulence is highly anisotropic. Indeed, a commonly observed feature of strongly stratified flows is the formation of quasi-horizontal layers, often described as pancake structures (e.g. Billant & Chomaz 2001). Such layers are conspicuous in simulations of tidal flows in strongly stratified fluids, both in non-rotating (Le Reun et al. 2018) and rotating fluids (Vidal et al. 2018). Hence, \(l_t\) depends on both the direction and the strength of the stratification. We introduce two turbulent length scales, respectively \(l_t^\parallel\) in the normal direction (i.e. along the gravity field) and \(l_t^\perp\) in the other horizontal directions.

Several regimes of stratified turbulence have been devised in fundamental fluid mechanics (Billant & Chomaz 2001; Brethouwer et al. 2007). They are characterised by the buoyancy Reynolds number

\[
R \sim \frac{\nu^3}{l_t^\parallel N_0^2},
\]

Le Reun et al. (2018) investigated the small-scale turbulence sustained by the tides in the regime \(R \leq 1\), in which vertical viscous shearing is important. However, radiative interiors are likely in the opposite regime \(R \gg 1\) (Mathis et al. 2018). Moreover, they neglected the rotation, by setting \(\Omega_s = 0\). In such a configuration, the subspace of waves \([H_\parallel, H_\perp]\) at the sub-harmonic resonance are empty, according to dispersion relations (15). Hence, the associated tidal instability involves only sub-harmonic resonances of internal waves \(H_\perp\) in the limit \(N_0/\Omega_s \to \infty\) and \(|\Omega_0| \to \infty\). Therefore, their results do not apply for our astrophysical problem, for any orbit in the range \(-1 \leq \Omega_0 \leq 3\). In the relevant strongly stratified regime \(R \gg 1\), diffusion is unimportant and the turbulence is three-dimensional (Brethouwer et al. 2007). The general scalings of this regime have been confirmed by turbulence simulations (e.g. Godeferd & Staquet 2003; Maffioli & Davidson 2016). Thus, they can be applied to the tidal problem.

In addition, rotational effects are also important within the orbital range \(-1 \leq \Omega_0 \leq 3\), even for large values of \(N_0/\Omega_s \geq 10\). Hence, the resulting turbulence undergoes the combined action of stratification and rotation.

In rotating stratified turbulence, the two turbulent length scales are related by (Billant & Chomaz 2001)

\[
l_t^\parallel \sim \frac{N_0}{\Omega_s} l_t^\perp,
\]

with \(\alpha_2 \sim 0.6\) a (numerical) pre-factor constrained from local turbulent simulations in rapidly rotating and strongly stratified turbulent regime (Reinaud et al. 2003; Waite & Bartello 2006). This regime is expected to be valid for radiative interiors, notably to describe shear-driven turbulence (Mathis et al. 2018). For strong stratifications \(N_0/\Omega_s \geq 10\), we combine the two balances obtained by equating (i) the nonlinear term with the buoyancy force in momentum equation (3a) and (ii) the injection term \((u \cdot \nabla)T_0\) and the nonlinear term \((u \cdot \nabla)\Theta\) in energy equation (3b). These balances yield respectively

\[
\frac{u^2}{l_t^\parallel} \sim \alpha_T g_0 \Theta_s \quad \text{and} \quad \alpha_T g_0 \Theta_s \sim N_0^2 l_t^\parallel,
\]

where \(\Theta_s\) is the typical dimensional turbulent buoyancy perturbation. We recover from balances (39) the classical scaling for the turbulent length scale in the normal direction, i.e. \(u_t \sim \frac{1}{\alpha_1} N_0\) (e.g. Billant & Chomaz 2001; Brethouwer et al. 2007). Hence, the turbulent length scale along the gravity direction is

\[
l_t^\parallel \sim \alpha_1 \beta_0 (1 - \frac{\Theta_s}{\Theta_0}) \frac{\Omega_s}{N_0} \quad \text{(with } \alpha_1 \sim 0.3 - 0.5\text{).}
\]

Scaling (40) shows that tidal mixing falls in the asymptotic regime of strongly stratified turbulence (Brethouwer et al. 2007). Then, we obtain two prescriptions for the eddy diffusivity, the first one valid in the gravity direction \(D_\parallel\) and the second one \(D_\perp\) in the perpendicular (horizontal) directions. They yield

\[
D_\parallel \propto \frac{1}{3} \frac{\alpha_2^2}{\beta_0} \frac{\Theta_0}{\Omega_0} (1 - \frac{\Theta_0}{\Theta_s}) \frac{\Delta l_t^\parallel}{l_t^\parallel} \quad \text{(41a)}
\]

\[
D_\perp \propto \frac{1}{3} \frac{\alpha_2}{\beta_0} \frac{\Theta_0}{\Omega_0} (1 - \frac{\Theta_0}{\Theta_s})^2 \quad \text{(41b)}
\]

with \(\alpha_1 \sim 0.3 - 0.5\) and \(\alpha_2 \sim 0.6\) (see above). Prescriptions (41) show that the eddy diffusivity should have a quadratic dependence with the equatorial ellipticity in any spatial direction. Another interesting prediction in this regime is that the turbulent potential and kinetic energies, defined by (in dimensional variables)

\[
E_\Theta(\Theta^*) \sim \frac{1}{2} \frac{\alpha_2^2}{\beta_0^2} \frac{\Theta_0^2}{N_0^2} \quad E_r(u^*) \sim \frac{1}{2} u^2,
\]

are of the same order of magnitude (Billant & Chomaz 2001). This can be checked in the numerical simulations (see later).

In-between the two aforementioned stratified regimes, i.e. when \(1 \leq N_0/\Omega_s \leq 10\), the situation is unclear. Indeed, Vidal et al. (2018) found that \(\mathbf{u} \cdot \mathbf{g}\), which is responsible for the normal tidal mixing, is largely unaffected by the stratification when \(N_0/\Omega_s \leq 10\) (see their figure 4). This would suggest to extend the prescription of the turbulent mixing (36) up to \(N_0/\Omega_s \leq 10\). Yet, this behaviour is not conspicuous in the numerics (see figure 9b of Vidal et al. 2018). This may be due to the rather specific numerical method, which inaccurately probed the intermediate regime \(1 \leq N_0/\Omega_s \leq 10\). Thus, we may expect also a transition between the two regimes (36) and (41) when \(1 \leq N_0/\Omega_s \leq 10\).

4.4. Validation against numerical simulations

We assess the relevance of predictions (36) and (41) by using direct numerical simulations. To do so, we solve in a global model the nonlinear and diffusive equations (3), which are supplemented by the stress-free conditions

\[
\mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{n} \times \left[(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)\mathbf{n}\right] = 0
\]

and assuming a fixed temperature \(\Theta = 0\). Stress-free conditions (43) are known to lead to spurious numerical behaviours, associated with the evolution of angular momentum in weakly deformed spheres (Guernond et al. 2013). To circumvent this numerical issue, we follow Cébron & Hollerbach (2014) and Vidal et al. (2018) by imposing a zero-angular momentum for the velocity perturbation. Moreover, the external region is assumed to be electrically insulating, such that the magnetic field \(\mathbf{b}\) matches a potential field at the boundary.

For the computations, we use the proof-of-concept global numerical model introduced in Vidal et al. (2018). Briefly, the reference ellipsoidal configuration (described in §2.3) is approximated in spherical geometry by an spatially varying equatorial ellipticity profile \(\epsilon(r, \beta_0)\), depending of the ellipticity \(\beta_0\).
of the ellipsoidal configuration. This profile is chosen such that the reference configuration satisfies all the aforementioned boundary conditions in the spherical geometry. The simulations have been performed with the open-source nonlinear code XSHELLS (https://nschaeff.bitbucket.io/xshells/), described in Schaeffer et al. (2017) and validated against standard spherical benchmarks (Marti et al. 2014; Matsui et al. 2016).

A second-order finite difference scheme is used in the radial direction. The angular directions are discretised using a pseudospectral spherical harmonic expansion, provided by the SHTns library (Schaeffer 2013). The time-stepping scheme is of second order in time and treats the diffusive terms implicitly, while the nonlinear and Coriolis terms are handled explicitly. We refer the reader to Vidal et al. (2018) for additional methodological details.

### The tidal problem

To estimate the global turbulent magnetic diffusivity (34), we measure the decay of an initial large-scale magnetic field (Yousef et al. 2003; Käpylä et al. 2019) in the presence of nonlinear tides, to compare it with the free decay rate of the same magnetic configuration in laminar diffusive models (e.g. Moffatt 1978). We compute the (dimensionless) decay rate $\sigma_\eta \leq 0$ of the volume average of the magnetic energy over the computational integration time $T$, i.e.

$$\sigma_\eta = \lim_{T \to \infty} \frac{1}{T} \log \left( \int_V \frac{1}{2} B^2 \, dV \right). \quad (44)$$

Decay rate (44) is a global estimate in the simulations of the effective diffusivity $D$. Note that Käpylä et al. (2019) measured in a similar way the turbulent diffusivity, obtaining a good quantitative agreement with mean-field analyses. Then, the global decay (44) rate should yield the same scaling law in $B_0$ for all the initial magnetic fields $B_0$, even if the (numerical) pre-factors will be different. Indeed, all the magnetic components will not obey the same scaling law in the strongly stratified regime (due to the anisotropic mixing). Notably, we expect toroidal magnetic fields, satisfying $B_0 \cdot \mathbf{1}_e = 0$ (at any position), to be preferentially mixed in the normal direction. Thus, scaling (41a) should apply predominantly for toroidal fields. On the contrary, we expect poloidal magnetic fields (with predominant components in the normal direction) to obey scaling (41b) in the horizontal directions. However, we emphasise that the pre-factors obtained from numerical simulations, performed for conditions far-removed from the astrophysical regimes, are likely irrelevant for the astrophysical problem (compared to the mixing-length predictions). Hence, we focus on the dependence in $B_0$, which should be generic whatever the topology of the initial magnetic field in the numerics. Thus, we aim at recovering (i) $\sigma_\eta \propto B_0$ in the weakly stratified regime (36) and (ii) $\sigma_\eta \propto B_0^2$ in the strongly stratified regime (41).

In magnetic radiative stars, the initial fossil field is unlikely force-free (e.g. Duez & Mathis 2010; Duez et al. 2010), except possibly close the stellar surface. The exact topology of the field relies on the Lorentz force, and only magnetic equilibria involving poloidal and toroidal components have been found (e.g. Braithwaite & Spruit 2017). Then, in addition to the slow laminar Joule diffusion, Braithwaite & Cantiello (2012) showed that an initial fossil field can decay due to the propagation of (slow) Magnetocoriolis waves (see appendix B) in the presence of rotation. Such a magnetic decay occurs on the (rather slow) dynamical time scale

$$\tau_{MC} \sim (\Omega_s L e^{-2})^{-1}. \quad (45)$$

Moreover, the field can be also dissipated by the turbulent mixing generated by the nonlinear tides. Thus, the initial field can be dissipated simultaneously by several mechanisms (neglecting in-situ dynamo mechanisms, regenerating the field against laminar and turbulent diffusion, which are highly debated).

However, we would like teh decay to be insensitive to the dynamical evolution (45) in the numerics, to investigate only the turbulent effects in a well controlled set-up. Hence, the aim is to find a magnetic configuration in which the initial field would decay solely by laminar Joule diffusion in the absence of tides. To do so, we can reasonably switch-off the Lorentz force in momentum equation, to estimate turbulent magnetic diffusivity (44) for a given initial magnetic field. Without the Lorentz force, MC waves are no longer sustained in the system. Moreover, as explained above, the Lorentz force surprisingly plays a negligible\footnote{Even though it is essential for the self-sustained generation of dynamo magnetic fields.} role on the turbulent mixing generated by the nonlinear tidal flows in the (relevant) weak field regime $L e \ll 1$ (Barker & Lithwick 2013b; Cébron & Hollerbach 2014; Vidal et al. 2018). Consequently, for this particular problem of the tidal instability, neglecting the Lorentz force is advisable in the numerics.

As a reference configuration, we have assumed $\Omega_0 = 0$. Indeed, we have shown theoretically in §3 that the underlying mechanism of the tidal instability does not change in the range $-1 \leq \Omega_0 \leq 3$, and similarly the turbulent scalings (e.g. Grannan et al. 2016; Vidal et al. 2018). Hence, investigating only one orbital configuration is necessary. Then, problem (33) reduces here to a kinematic (i.e. linear) initial value problem for the initial field. We emphasise that the exact topology of the initial field will not be essential here for the numerical model. Indeed, without Lorentz force, induction equation (33a) is uncoupled to the momentum equation. To mimic the slow magnetic decay on the laminar Joule diffusion (in the absence of tides), we have chosen for the initial fossil field the least-damped, poloidal free decay magnetic mode of the sphere (see Moffatt 1978, p. 36-40). This particular magnetic field is an exact solution of the purely diffusive induction equation. It has the smallest laminar Ohmic free decay rate $\sigma_\Omega$ (in dimensionless form), given by

$$\sigma_\Omega = \pi^2 E k / Pm. \quad (46)$$

Thus, this is the most suited initial magnetic field to assess the validity of the turbulent scaling laws. Indeed, the slow laminar Joule diffusion (46) should not be coupled with the expected faster turbulent diffusion in the numerics to get robust results. In practice, we conducted the simulations at the fixed dimensionless numbers $Ek = 10^{-4}$, $Pr = 1$ and $Pm = 0.1$. The latter value ensures that no dynamo magnetic field can grow exponentially. Our spatial discretisation is $N_r = 224$ radial points, $l_{\text{max}} = 128$ spherical harmonic degrees and $m_{\text{max}} = 100$ azimuthal wave numbers. We have integrated the equations on one (dimensionless) Ohmic diffusive time $(E k / Pm)^{-1}$, to determine accurately the turbulent decay rate $\sigma_\eta$. Figure 8 shows the representative results for the two stratified regimes. We observe that the decay rate $\sigma_\Omega$ is always larger than the free decay rate $\sigma_\eta$ expected for the initial fossil field. Then, the striking feature is that we recover the two scalings as a function of the ellipticity, predicted by our mixing-length theory. In the weakly stratified regime (figure 8a), numerical decay (44) is in broad agreement with the linear scaling $\sigma_\eta \propto B_0$, consistent with mixing-length formula (36). The agreement is even much better in the strongly stratified regime (figure 8b), obtaining the quadratic scaling $\sigma_\eta \propto B_0^2$ expected from (41).

We have found that the observed enhancement generated by the tides is rather weak in the simulations. This is not due to...
the tidal amplitude, which is already two orders of magnitude larger than typical values for binaries (i.e. \( \beta_0 \approx 10^{-1} \) in the numerics compared to \( \beta_0 \approx 10^{-3} - 10^{-2} \), see Table 2 below). This simply comes from the over-estimated value of the laminar Joule diffusion in the simulations (i.e. \( \text{Ek} / \text{Pm} = 10^{-3} \)). This makes the laminar and turbulent decay rates roughly comparable in amplitude. Simulations in the astrophysical regime (i.e. \( \text{Ek} / \text{Pm} \lesssim 10^{-10} \)) would be more affected by the tides. Yet, our simulations always support the trend predicted by the mixing-length theory (41). For stellar conditions, the latter predicts that the tidal decay rate would be much stronger than the laminar Joule decay rate (see the discussion in §5).

Finally, the typical ratio of the volume averaged thermal and kinetic (dimensionless) energies, for the simulations of figure 8b, is \( E(\Theta) / E(u) = 8.1 \pm 3.5 \). This numerical value is in very good agreement with the theoretical scaling (42) in the strongly stratified regime (Billant & Chomaz 2001), yielding \( E(\Theta) / E(u) \sim N_0/\Omega_s = 10 \) in dimensionless variables. This is another evidence of the validity of the mixing-length theory.

5. Astrophysical discussion

We have obtained a consistent picture of the tidal instability in an idealised set-up of radiative interiors. It predicts the linear onset (§3) and the nonlinear mixing induced by the saturated flows (§4). For the sake of theoretical and numerical validations, we have only considered rather idealised stellar models, described in §2. Then, the predictions have been successfully confronted with proof-of-concept numerical simulations, paving the way for astrophysical applications.

Indeed, we emphasise that the theory can a priori embrace more realistic stellar conditions. In particular, the mixing-length theory is only based on local dimensional arguments, that should remain valid for more realistic conditions. Therefore, we discuss now our findings in the context of tidally deformed and stably stratified (radiative) interiors. Notably, we are in the position to build a new physical scenario, that may explain the lower incidence of fossil fields in some short-period and non-synchronised binaries (Alecian et al. 2017).

5.1. A new scenario?

We consider a close binary system with a radiative primary of mass \( M_1 \) and a secondary of mass \( M_2 \). The primary is pervaded by an initial fossil field \( \mathbf{B}_0 \). Note that distinction between the primary and secondary is only made for convenience, such that the situation can be reversed in the scenario (if we are interested in the secondary). The orbital and spin angular velocities are respectively \( \Omega_{orb} \) and \( \Omega_s \). We focus on non-synchronised binaries in the orbital range \( -1 \leq \Omega_s / \Omega_0 \leq 3 \), where \( \Omega_0 = \Omega_{orb} / \Omega_s \) is the dimensionless orbital frequency. The orbits are almost circularised, but small orbital eccentricities \( \epsilon \ll 1 \) do not strongly modify the fate of tidal flows (Vidal & Cébron 2017). We also focus on binaries with short-period systems, with typical periods \( T_s = 2\pi / \Omega_s \lesssim 10 \) days. Due to the combined action of the tides and the spin, the star is deformed into an triaxial ellipsoid (Chandrasekhar 1969; Lai et al. 1993; Barker et al. 2016). The latter is characterised by a typical equatorial ellipticity \( \beta_0 \) estimated from the static bulge theory (Cébron et al. 2012b; Vidal et al. 2018). For the bulge generated onto the primary, this reads

\[
\beta_0 \sim \frac{3}{2} \frac{M_2}{M_1} \left( \frac{R}{D} \right)^3,
\]

where \( R \) is the typical radius of the primary and \( D \) the typical distance separating the two bodies. The density stratification of the radiative envelope is measured by the typical dimensionless ratio \( N_0 / \Omega_0 \), where \( N_0 \) is the typical Brunt-Väisälä frequency. A representative value for intermediate-mass stars is \( N_0 \sim 10^{-3} \text{s}^{-1} \) (e.g. Rieutord 2006), yielding a typical ratio \( N_0 / \Omega_0 \gg 10 \).

The tidal forcing sustains an equilibrium tidal velocity field (Remus et al. 2012; Vidal & Cébron 2017) in the primary fluid body. This equilibrium tidal flow can be nonlinearly coupled with inertial-gravity waves, triggering tidal instabilities. The dimensional growth rate \( \sigma^* \) of the tidal instabilities, which does not depend on the stratification, is given by

\[
\sigma^* = \frac{(2\Omega_0 + 3)^2}{16(1 + \Omega_0^2)} \frac{\Omega_s - \Omega_{orb}}{\Omega_s} |\beta_0|,
\]
with \( \Omega_0 = \Omega_0/(1 - \Omega_0) \). In the saturated regime, the tidal instabilities enhance the internal mixing due to turbulence. In strongly stratified radiative interiors \((N_0/\Omega_s \gg 10)\), the turbulent mixing generated by the nonlinear tides is anisotropic, characterised by an eddy turbulent diffusivity \( D^\perp \) in the direction of the self-gravity and by \( D^\parallel (\gg D^\perp) \) in the other (horizontal) directions.

Then, the turbulent mixing will dynamically enhance the Joule decay of the fossil field \( \mathbf{B}_0 \). However, the latter field, containing both poloidal and toroidal components (to be in quasi-static magnetic equilibrium in the initial stage), will undergo an enhanced anisotropic turbulent Joule diffusion. The mechanism is illustrated in figure 9. On the one hand, the poloidal components, which are mainly along the normal direction and can be observed at the stellar surface, would be preferentially dissipated by the (large) eddy diffusivity \( D^\perp \) in the horizontal directions. On the other hand, the toroidal components, hidden in the star since they are only along the horizontal directions, are preferentially mixed by the (small) eddy diffusivity \( D^\parallel \) in the normal direction. Thus, poloidal and toroidal field lines are dissipated on different turbulent time scales. This would yield a global dissipation of the magnetic equilibrium within the stellar interior (at the position \( r \leq R \)) on a few typical time scales \( \tau_1 \) given by

\[
\tau_1 \propto \frac{r_1^2}{D_1^\perp} \sim \frac{K_\alpha}{\beta_0^3 \Omega_0 (1 - \Omega_0)^2}.
\]

with the pre-factor \( K_\alpha \sim 30 - 50 \) estimated from formulas (41).

Time scale (49) is the (fast) turbulent time scale in the perpendicular (horizontal) directions. In addition, the magnetic field would also die out in the presence of rotation on the dynamical time scale of (slow) Magneto-Coriolis waves (45), as shown by Braithwaite & Cantiello (2012).

5.2. Non-magnetic binaries

We assess here the relevance of the tidal scenario for short-period massive binary systems. Non-magnetic and non-synchronised \((\Omega_0 \neq 1)\) binaries are given in table 2. They have been surveyed by the BinaMiCS collaboration (e.g. Alecian et al. 2017). The predictions of the tidal scenario for these binary systems are given in table 3. All these close-binaries are rapidly rotating and undergo strong tidal effects in the two bodies, as measured by the large values of the ellipticity \( \beta_0 \sim 10^{-2} \). Hence, the strong tides should trigger quickly the tidal instability, growing on the typical time scale \((\sigma^*)^{-1} \approx O(10^3) \) years. This is much shorter than the lifetime of these stars, about \( \tau_{\text{MS}} \sim 10^9 \) years for a star of mass \( M_1 = 2M_\odot \) on the main sequence. Hence, the tidal instability is likely present in these non-synchronised binaries.

Then, typical values for the turbulent time scale (49) are \( \tau_1 \in [10^3, 10^7] \) years, except for HD 23642 and HD 32964 which are less affected by the tidal instability (smaller \( \beta_0 \)). For the favourable systems, the turbulent Joule diffusion of the initial fossil fields may occur on very short time scales compared to the stellar lifetime, typically \( \tau_1/\tau_{\text{MS}} \ll 10^{-3} \) for the more favourable systems. This turbulent time scale (49) is often smaller that the dynamical time scale \( \tau_{\text{MC}} \) put forward by Braithwaite & Cantiello (2012), given by expression (45).

Therefore, nonlinear tidal flows in non-synchronised may sustain an enhanced turbulent Joule diffusion of fossil fields, occurring on time scales which are often shorter than the stellar lifetime. This may explain the scarcity of significant magnetic fields at the surface of some massive stars in short-period binaries.

5.3. Magnetic binaries

We give in table 4 the orbital properties of some scarce magnetic binaries, analysed by the BinaMiCS collaboration. They were al-
The fate of the tidal instability for synchronised orbits ($\Omega = \omega$) is given by formula (49). The time scale of turbulent Joule diffusion $\tau$ is given by formula (49) with $K_0 = 30$. The laminar Ohmic diffusive time scale is $\tau_\Omega = (\Omega \Delta k / P)^{-1}$ (in dimensional units of $\Omega_0$) with $E \Delta k / P \sim 10^{-12}$. The dynamical time scale associated with the propagation of (slow) Magneto-Coriolis waves is $\tau_{MC} = (\Omega_0 / \tau_\Omega)^{-1}$ (Braithwaite & Cantillo 2012), with $L_e \sim 10^3$.

<table>
<thead>
<tr>
<th>System</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$D$</th>
<th>$T_\text{orb}$</th>
<th>$T_\text{orb}$</th>
<th>$e$</th>
<th>$\beta_0$</th>
<th>Body 1</th>
<th>Body 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>HD 23642</td>
<td>1.03 $\times 10^{-4}$</td>
<td>5.07 $\times 10^{-3}$</td>
<td>1.58 $\times 10^{-3}$</td>
<td>6.88 $\times 10^{-3}$</td>
<td>1.46 $\times 10^{-2}$</td>
<td>6.44 $\times 10^{-2}$</td>
<td>1.5 $\times 10^{-2}$</td>
<td>6.4 $\times 10^{-2}$</td>
<td></td>
<td></td>
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<tr>
<td>HD 24133</td>
<td>1.46 $\times 10^{-3}$</td>
<td>6.11 $\times 10^{-3}$</td>
<td>2.26 $\times 10^{-3}$</td>
<td>1.53 $\times 10^{-3}$</td>
<td>6.26 $\times 10^{-4}$</td>
<td>4.48 $\times 10^{-4}$</td>
<td>6.3 $\times 10^{-4}$</td>
<td>4.5 $\times 10^{-4}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HD 24099</td>
<td>9.25 $\times 10^{-3}$</td>
<td>9.26 $\times 10^{-3}$</td>
<td>2.61 $\times 10^{-3}$</td>
<td>3.24 $\times 10^{-3}$</td>
<td>3.3 $\times 10^{-4}$</td>
<td>3.3 $\times 10^{-4}$</td>
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<td></td>
</tr>
<tr>
<td>HD 25638</td>
<td>1.12 $\times 10^{-3}$</td>
<td>2.03 $\times 10^{-3}$</td>
<td>9.89 $\times 10^{-4}$</td>
<td>3.6 $\times 10^{-4}$</td>
<td>7.89 $\times 10^{-4}$</td>
<td>2.99 $\times 10^{-4}$</td>
<td>6.8 $\times 10^{-4}$</td>
<td>3.0 $\times 10^{-4}$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>HD 25833</td>
<td>4.78 $\times 10^{-3}$</td>
<td>5.83 $\times 10^{-3}$</td>
<td>2.06 $\times 10^{-4}$</td>
<td>1.7 $\times 10^{-4}$</td>
<td>1.1 $\times 10^{-3}$</td>
<td>1.7 $\times 10^{-3}$</td>
<td>1.1 $\times 10^{-3}$</td>
<td>1.4 $\times 10^{-3}$</td>
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<tr>
<td>HD 32964</td>
<td>7.48 $\times 10^{-4}$</td>
<td>4.83 $\times 10^{-3}$</td>
<td>1.22 $\times 10^{-3}$</td>
<td>3.61 $\times 10^{-3}$</td>
<td>5.01 $\times 10^{-3}$</td>
<td>1.49 $\times 10^{-3}$</td>
<td>5.0 $\times 10^{-3}$</td>
<td>1.5 $\times 10^{-3}$</td>
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<td></td>
</tr>
<tr>
<td>HD 34364</td>
<td>1.12 $\times 10^{-2}$</td>
<td>7.12 $\times 10^{-3}$</td>
<td>9.53 $\times 10^{-3}$</td>
<td>2.25 $\times 10^{-3}$</td>
<td>5.01 $\times 10^{-3}$</td>
<td>1.1 $\times 10^{-3}$</td>
<td>1.2 $\times 10^{-3}$</td>
<td>1.3 $\times 10^{-3}$</td>
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<td></td>
</tr>
<tr>
<td>HD 36486</td>
<td>2.20 $\times 10^{-3}$</td>
<td>4.32 $\times 10^{-3}$</td>
<td>1.18 $\times 10^{-3}$</td>
<td>3.22 $\times 10^{-3}$</td>
<td>4.35 $\times 10^{-4}$</td>
<td>3.47 $\times 10^{-4}$</td>
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<td>HD 150136</td>
<td>5.98 $\times 10^{-1}$</td>
<td>7.53 $\times 10^{-2}$</td>
<td>3.49 $\times 10^{-2}$</td>
<td>7.37 $\times 10^{-2}$</td>
<td>2.76 $\times 10^{-2}$</td>
<td>2.66 $\times 10^{-2}$</td>
<td>2.8 $\times 10^{-2}$</td>
<td>6.3 $\times 10^{-2}$</td>
<td></td>
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</tr>
</tbody>
</table>

Table 4: Physical and orbital characteristics of the magnetic binary systems surveyed by the BinaMicS collaboration (Folsom et al. 2013; Shultz et al. 2015, 2017, 2018). The masses $[M_1, M_2]$ of the primary and the secondary bodies are given in Sun mass unit $M_0$. The typical stellar radius $R$ (of either the primary or of the secondary) and the typical distance $D$ between the two bodies is given in Sun radius unit $R_0$. The spin and orbital periods $[T, T_{\text{orb}}]$ are expressed in days. They yield the spin and angular velocities $[\Omega = 2\pi / T, \omega_{\text{orb}} = 2\pi / T_{\text{orb}}]$. The typical surface magnetic field $B_{\text{surf}}$ believed to be of fossil origin, is given in kiloGauss (kG) for the two components. HD 156324 and HD 98088 are synchronised systems (see appendix D), whereas $e$ Lupi system is not synchronised.

<table>
<thead>
<tr>
<th>System</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$D$</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$T_{\text{orb}}$</th>
<th>Eccentricity</th>
<th>$B_{\text{surf}}$ ($M_1$)</th>
<th>$B_{\text{surf}}$ ($M_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>HD 156324</td>
<td>8.5</td>
<td>4.1</td>
<td>3.8</td>
<td>2.3</td>
<td>13.2</td>
<td>1.58</td>
<td>1.58</td>
<td>0.0</td>
<td>14</td>
<td>2.6</td>
<td>0.9</td>
</tr>
<tr>
<td>HD 98088</td>
<td>2.19</td>
<td>1.67</td>
<td>2.76</td>
<td>1.77</td>
<td>21.7</td>
<td>5.905</td>
<td>5.905</td>
<td>5.905</td>
<td>0.18</td>
<td>3.9</td>
<td>1.6</td>
</tr>
</tbody>
</table>

We note that HD 156324 and HD 98088 are synchronised. The fate of the tidal instability for synchronised orbits ($\Omega = 1$) is discussed in appendix D. On the one hand, system HD 156324 is nearly circularised (Shultz et al. 2017), whereas non-circular orbits are required for the tidal mechanism to operate in synchronised systems. Hence, the tidal mechanism is not currently relevant for HD 156324. This may explain why the fossil field is ready known to be magnetic, e.g. HD 98088 (Babcock 1958; Abt et al. 1968; Carrié et al. 2002), $e$ Lupi (Shultz et al. 2015) and HD 156324 ( Alecian et al. 2014a). The aforementioned tidal scenario would suggest that (strong) magnetic fields may be anomalies in short-period massive binaries. However, their existence does not necessarily challenge the tidal scenario (a priori).
is still observed. On the other hand, HD 98088 is not car- 

ularised such that nonlinear tidal mixing is expected. However, as shown in appendix D, formula (49) for the typical turbulent time scale ought to be reduced in synchronised systems, such that 

\[(1 - \Omega_0)^2 \sim \epsilon^2 \]

where \(\epsilon \ll 2\epsilon\) is the dimensionless amplitude of the differential rotation on the elliptical orbit (Cébron et al. 2012b; Vidal & Cébron 2017). Based on the accuracy of the measured periods in table 4, we may assume \(\epsilon \leq 10^{-3}\), such that the turbulent time scale \(\tau_\epsilon\), given by formula (D.9), is expected to be much larger in HD 98088 than for the systems of table 3 (for similar values of the equatorial ellipticity \(\beta_0 \sim 10^{-3}\)). Therefore, the existence of the (synchronised) magnetic binaries HD 156324 and HD 98088 appears to be consistent with the tidal scenario. However, note that the tidal mechanism may have occurred before the synchronisation and/or circularisation of the systems. Indeed, observations show that circularisation and synchronisation pro- cesses are effective for radiative stars (e.g. Giuricin et al. 1984b,a; Zimmerman et al. 2017). On the other hand, circularisation is ex- pected to be faster in radiative stars, e.g. due to radiative damping (e.g. Zahn 1975, 1977). On the other hand, synchronisation pro- 

cesses have been much less studied in radiative interiors (e.g. Rocca 1989, 1987; Witte & Savonije 1999, 2001), and the con- 

frontation with the observations is less satisfactory (e.g. Mazeh 2008; Zimmerman et al. 2017). Thus, understanding these two processes in radiative stars still deserves further work, notably to take into account the overlooked effects of the tidal instability in short-period binaries.

Finally, the case of \(\epsilon\) Lupi system (e.g. Uytterhoeven et al. 2005; Shultz et al. 2015) is more intricate. Nonlinear tidal mixing should occur within these stars, with a typical turbulent time scale \(\tau_\epsilon \sim 10^3\) years. The fossil field may be currently dissipated by the nonlinear tides, but the process may have not last long enough to yield vanishing observable fields. Another possibility is that these magnetic fields would be internally regenerated by dynamo action, to balance the decay due to the nonlinear tides. Such a (currently speculative) mechanism may be particularly relevant for the rapidly rotating component of \(\epsilon\) Lupi in table 4. Several dynamo mechanisms may be advocated, e.g. driven by differentially rotating flows (Braithwaite 2006), baroclinic flows (Simitev & Busse 2017) or even nonlinear tides (Vidal et al. 2018). Though the dynamo action of tides in strongly stratified interiors remains elusive, the scaling law for the magnetic field strength at the stellar surface, proposed by Vidal et al. (2018), would yield \(|B_0| \sim 0.1 - 1\) kG. This is the order of magnitude of the observed surface fields. Thus, understanding the origin of the magnetic fields in the \(\epsilon\) Lupi system certainly deserves future studies.

6. Conclusion

6.1. Summary

In this work, we have investigated nonlinear tides in short-period massive binaries, motivated by the puzzling lower magnetic in- 
cidence of close binaries compared to isolated stars (Alecian et al. 2017; Alecian et al. 2019). To do so, we have adopted an idealised model for rapidly rotating stratified fluids within the Boussinesq approximation. This model consistently takes into ac- 

count all the ingredients encountered in massive binaries, namely the combination of rotation and non-isentropic stratification, the tidal distortion (on coplanar and aligned orbits) and the leading- 

order magnetic effects. We have revisited the fluid instabilities triggered by the nonlinear tides in the system (Vidal et al. 2018), by combining analytical computations and proof-of-concept sim- 

ulations.

First, we have studied the linear onset of the tidal instability in non-synchronised, stratified fluid masses. Within a single frame- 

work, we have unified all the previous existing stability analyses and we have unravelled new phenomena. We have shown that the tidal instability in radiative stratified interiors is due to parametric 

resonances between inertial-gravity waves and the underlying equilibrium tidal flow, for any orbit in the range \(-1 \leq \Omega_0 \leq 3\). Within this orbital range, the tidal instability is weakened by a barotropic stratification on the polar axis (Miyazaki & Fukumoto 1991; Miyazaki 1993) and in the equatorial plane. On the contrary, a baroclinic stratification does enhance the growth rate of the tidal instability (Kerswell 1993a; Le Bars & Le Dizés 2006). However, the striking feature is that the tidal instability 

onsets with a maximum growth rate which is unaffected by the stratification. The instability is triggered in volume along three- 

dimensional conical layers, whose position depends solely on the orbital parameter \(\Omega_0\). In the other orbital range \(\Omega_0 \leq -1\) and \(\Omega_0 \geq 3\), i.e. in the forbidden zone of the tidal instability in ho- 

mogeneous fluids (e.g. Le Dizès 2000), the tidal instability can be generated by parametric resonances of gravito-inertial waves, 

provided that the stratification is strong enough for the considered orbital configuration. This provides a theoretical explanation of the instability mechanism investigated numerically in Le Reun et al. (2018).

Second, we have developed a mixing-length theory (e.g. Tennekes & Lumley 1972) of the anisotropic turbulent mixing, 
sustained by the nonlinear tidal flows in the orbital regime \(-1 \leq \Omega_0 \leq 3\). For strongly stratified interiors, we have mod- 

eled the anisotropic turbulent mixing (e.g. Mathis et al. 2018) by introducing two turbulent eddy diffusivities, one describing the mixing in the direction of the gravity field and the second in the other (horizontal) directions. We have shown that these two turbulent diffusivities should scale as \(\beta_0\), where \(\beta_0\) is the equatorial ellipticity of the equilibrium tide. We have assessed these scalings against proof-of-concept simulations, by using the numerical method introduced in Vidal et al. (2018).

Finally, we have used the mixing-length theory to extrapolate the numerical results towards more realistic stellar conditions. We have built a new physical scenario, predicting an enhanced Joule diffusion of fossil fields due to turbulent mixing induced by the nonlinear tidal flows in short-period (non-coalescing) massive binaries. We have applied it to a subset of short-period binaries, 

analysed by the BinaMIcS collaboration (Alecian et al. 2017; Alecian et al. 2019). This scenario may (partially) explain the lower incidence of surface magnetic fields in some short- 

period binaries (compared to isolated stars). Indeed, we predict a turbulent Joule diffusion of fossil fields occurring in a few million years for the most favourable systems. This is much shorter than the (laminar) Joule diffusion time scale of the fossil fields, and similarly than the typical lifetime of these stars. Therefore, we cannot rule out a priori the tidal mechanism to explain the scarcity of massive magnetic stars in close binary systems.

6.2. Perspectives

We have shown that the tidal mechanism is plausible, because close binaries are known to be strongly deformed by tides. Then, future studies should strive to assess the likelihood of this new mechanism with more realistic physical models. In- 

deed, we have only handled the key physical ingredients. Hence, many improvements are worth doing on the numerical and theo- 

retical fronts.
First, the validity of mixing-length models for the magnetic diffusivity is questionable. Though they are commonly used in hydromagnetic turbulence (e.g. Yousef et al. 2003; Käpylä et al. 2019), Vainshtein & Rosner (1991) proposed that even weak large-scale magnetic fields may suppress the turbulent magnetic diffusion. This behaviour has been obtained in simulations of non-rotating, two-dimensional turbulence (e.g. Cattaneo & Vainshtein 1991; Cattaneo 1994; Kondić et al. 2016). However, the relevance of this inhibiting mechanism for three-dimensional, rotating and tidally driven turbulence remains unclear, e.g. because Alfvén waves do not play (a priori) a significant role in the tidal turbulent mixing (compared to inertial waves). Indeed, it seems in contradiction with the turbulent hydromagnetic simulations of Barker & Lithwick (2013b), who showed that a weak magnetic field can rather enhance the small-scale tidal turbulence. Thus, investigating this effect in tidally forced turbulence seems necessary, by performing demanding simulations of the consistent rotating hydromagnetic set-up.

Second, it would be interesting to examine if (secondary) shear instabilities are sustained by the nonlinear tides in the strongly stratified regime. Shear instabilities are likely ubiquitous in radiative interiors (e.g. Mathis et al. 2004, 2018), which undergo differential rotations (Goldreich & Schubert 1967). To do so, the usual diffusionless instability condition for shear instabilities (e.g. Drazin & Reid 1981) ought to be modified in radiative interiors, to take into account the thermal diffusivity (Townsend 1958; Zahn 1974). Hence, in the presence of turbulent tidally driven flows, secondary shear instabilities may exist if

$$Ri \times Pe_\parallel \leq 1,$$

(50)

with $Ri = N_l^2 / (u_l l_i)^2$ the turbulent Richardson number and $Pe_\parallel = u_l l_i / D_\parallel^2$ the turbulent Péclet number. By using our mixing-length predictions, a typical estimate would be $Ri \times Pe_\parallel \sim 1$ in the strongly stratified regime. Thus, such secondary shear instabilities might be triggered by the nonlinear tidal flows. This may enhance further the turbulent diffusion coefficients.

Then, a natural extension would be to investigate consistently the interplay between the nonlinear tides and differential rotation, e.g. resulting from in-situ baroclinic torques (e.g. Busse 1981, 1982; Rieutord 2006). Whether differential rotation is important for the tidal mixing is elusive, e.g. because differential rotation is damped by several hydromagnetic effects (Moss 1992; Spruit 1999; Arlt et al. 2003; Rüdiger et al. 2013, 2015; Jouve et al. 2015). Nonetheless, the tidal elliptical instability does exist in differentially rotating elliptical flows, as shown in fundamental fluid mechanics (Eloy & Le Dizès 1999; Lacaze et al. 2007). The properties of the waves for more astrophysically relevant differential rotation profiles can be investigated in global models (Friedlander 1989; Mirouh et al. 2016), such that extending the present theory seems achievable. Closely related to the study of differential rotation is the study of baroclinic flows (e.g. Kitchatinov 2014; Caleo & Balbus 2016; Simitev & Busse 2017). We have shown that baroclinic reference states do enhance the tidal instability, as first noticed by Kerswell (1993a) and Le Bars & Le Dizès (2006). Thus, we may even expect a stronger turbulent tidal mixing in baroclinic radiative interiors.

Radiative stars also host innermost convective cores. Thus, the outcome of the tidal instability in shells should be considered. The tidal (elliptical) instability does exist in shells, as confirmed experimentally and numerically for homogeneous fluids (Aldridge et al. 1997; Seyed-Mahmoud et al. 2000; Lacaze et al. 2005; Seyed-Mahmoud et al. 2004; Lemasquerier et al. 2017). Indeed, the local stability theory we have presented remains formally valid in shells. Hence, we do not expect any significant difference for stratified fluids at the onset. Then, the boundary effects on the turbulent tidal mixing remain to be determined.

Another daunting perspective is to account for compressibility. Using the Boussinesq approximation seems exaggerated in global models of stellar interiors. However, the influence of the fluid compressibility is apparently negligible at the onset of the tidal instability (Clausen & Tilgner 2014). This is one of the reasons why we have adopted the Boussinesq approximation. Then, investigating non-isentropic reference profiles cannot be easily simulated numerically with the more usual anelastic approximation (e.g. Anufriev et al. 2005). Moreover, our mixing-length theory only invokes local estimates. In particular, we may naively expect the radial mixing to be only governed by the local value of the stratification, independently of its origin. Moreover, the compressibility would barely modify the (strongest) horizontal mixing, since horizontal motions are less inhibited by compressibility. Therefore, our typical turbulent time scale may still be relevant in compressible interiors. Nonetheless, clarifying the effects of compressibility deserves future works, both in the linear and nonlinear regimes.

Finally, the scarce non-synchronised magnetic binaries (Carrier et al. 2002; Shultz et al. 2015; Alecian et al. 2017; Kochukhov et al. 2018) seem to challenge the general trend of the tidal scenario, predicting a lack of magnetic massive stars in short-period binaries. These fields do not appear to be strongly dissipated by the nonlinear tidal flows. If the tidal mechanism remains valid by including the aforementioned proposed improvements, they might be dynamically regenerated in situ by dynamo action. For instance, tides do sustain dynamo action at small-scale (Barker & Lithwick 2013b) and large-scale (Cébron & Hollerbach 2014; Reddy et al. 2018) in homogeneous fluids, as well as in weakly stratified interiors (Vidal et al. 2018). Yet, the dynamo capability of tides remains elusive in strongly stratified interiors (Vidal et al. 2018). Baroclinic flows are another possible candidate, since they are dynamo capable (Simitev & Busse 2017). They may also favour the radial mixing generated by nonlinear tides, which is a necessary ingredient for dynamo action (Kaiser & Busse 2017). This certainly deserves future works to investigate dynamo magnetic fields in more realistic models of radiative stars.

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Appendix A: Local (WKB) stability equations

We present the local Wentzel-Kramers-Brillouin (WKB) stability method. In the local analysis, the unbounded growth of the perturbations gives sufficient conditions for local instability (Friedlander & Vishik 1991; Lifschitz & Hameiri 1991). The original WKB hydrodynamic stability theory has been extended by several authors, e.g. to take into account buoyancy effects within the Boussinesq approximation (Kirillov & Mutabazi 2017).

In the following, we derive the coupled (WKB) stability equations for arbitrary, spatially varying Boussinesq and magnetic background states. We emphasise that their derivation is intrinsically different from the one of Kelvin wave stability equations (Craik & Criminale 1986; Craik 1989), also accounting for magnetic fields (Craik 1988; Fabijonas 2002; Lebovitz & Zweibel 2004; Herreman et al. 2009; Mizerski & Bajer 2011; Cébron et al. 2012b; Mizerski et al. 2012; Mizerski & Lyra 2012; Bajer & Mizerski 2013) and buoyancy effects (Cébron et al. 2012b). Indeed, the Kelvin wave method cannot investigate the stability of arbitrary background states, contrary to the WKB method.

Appendix A.1: Linearised stability equations

We use in the following dimensional variables to devise the general stability equations in the diffusionless limit. Contrary to the main text, the dimensional variables are written here without \( \rho \), to keep concise mathematical expressions. We consider a fluid rotating at the angular velocity \( \Omega \) and stratified in density under a spatially varying background state \( [U_0, B_0, \Theta_0] \). In unbounded fluids, the perturbations are governed by the linearised hydromagnetic, Boussinesq equations

\[
\begin{align*}
\frac{du}{dt} &= -(u \cdot \nabla) U_0 - 2 \Omega \times u - \nabla(p + p_b) - \alpha_T \Theta g + \alpha_B (B_0 \cdot \nabla b + b \cdot \nabla B_0), \\
\frac{db}{dt} &= b \cdot \nabla U_0 - (u \cdot \nabla) B_0 + (B_0 \cdot \nabla) u, \\
\frac{d\Theta}{dt} &= -(u \cdot \nabla) \Theta_0, \\
\nabla \cdot u &= 0, \quad \nabla \cdot b = 0,
\end{align*}
\]

where \( \frac{d}{dt} = \partial / \partial t + (U_0 \cdot \nabla) \) is the material derivative along the basic flow, \( p \) is the hydrodynamic pressure and \( p_b = \alpha_B (B_0 \cdot b) \) the magnetic pressure. In equations (A.1), \( \alpha_T \) is the coefficient of thermal expansion (at constant pressure) in the Boussinesq equation of state (EoS)

\[
\delta \rho = \rho_M (1 - \alpha_T \Theta),
\]

with \( \delta \rho \) the Eulerian perturbation in density.

Appendix A.2: Short-wavelength perturbations

We seek short-wavelength perturbations in Eulerian description, with respect to the small asymptotic parameter \( 0 < \varepsilon \ll 1 \). We introduce the formal asymptotic series

\[
\begin{align*}
u(r, t) &= [u^{(0)} + \varepsilon u^{(1)}] (r, t) \exp(i \Phi(r, t) / \varepsilon) + \ldots, \\
b(r, t) &= [b^{(0)} + \varepsilon b^{(1)}] (r, t) \exp(i \Phi(r, t) / \varepsilon) + \ldots, \\
\Theta(r, t) &= [\Theta^{(0)} + \varepsilon \Theta^{(1)}] (r, t) \exp(i \Phi(r, t) / \varepsilon) + \ldots, \\
p(r, t) &= [p^{(0)} + \varepsilon p^{(1)}] (r, t) \exp(i \Phi(r, t) / \varepsilon) + \ldots,
\end{align*}
\]

where \( \Phi \) is a real-valued scalar function that represents the rapidly varying phase of oscillations and \( [u^{(0)}, \Theta^{(0)}, p^{(0)}] \) are slowly varying complex-valued amplitudes. Note that we have omitted in expansions (A.3) the reminder terms, assumed to be uniformly bounded in \( \varepsilon \) on any fixed time interval (Lifschitz & Hameiri 1991; Lebovitz & Lifschitz 1992; Lifschitz & Lebovitz 1993).

We further introduce the local wave vector, defined by \( k = \nabla \Phi \).

The small asymptotic parameter \( \varepsilon \ll 1 \) is actually related to the typical scale of the instability \( L \), which must be much smaller to the typical length scale of the large-scale background flow \( L_0 \). This yields (Nazarenko et al. 1999) \( \varepsilon = l / L_0 \ll 1 \). In the hydrodynamic and diffusionless case, its value is arbitrary small.

However, in hydromagnetics, \( \varepsilon \) does affect the magnetic field, since the Lorentz force depends on the length scale. The general magnetic configuration leads to a set of partial differential equations (Friedlander & Vishik 1995; Kirillov et al. 2014), which must be solved locally in Eulerian description. However, by assuming (see also for uniform fields Mizerski & Bajer 2011)

\[
B_0(r) = \varepsilon \hat{B}_0(r),
\]

the partial differential equations simplify into ordinary differential equations (even for spatially varying magnetic fields). This is the central approximation of the hydromagnetic stability theory, which is not required in the non-magnetic case. For tidal studies, we usually set \( \varepsilon = \beta_0 \) (Le Dizès 2000).

Appendix A.3: Eulerian stability equations

We closely follow the mathematical derivation of Kirillov & Mutabazi (2017), extending it to the hydromagnetic case. Substituting expansions (A.3) in incompressible condition (A.1d) and collecting terms of order \( i / \varepsilon^{-1} \) and \( \varepsilon^0 \) yields

\[
\begin{align*}
i / \varepsilon^{-1} & : \quad \begin{bmatrix} u^{(0)}, b^{(0)} \end{bmatrix} \cdot k = 0, \quad \varepsilon^0 : \quad \nabla \cdot \begin{bmatrix} u^{(0)}, b^{(0)} \end{bmatrix} = -i k \cdot \begin{bmatrix} u^{(1)}, b^{(1)} \end{bmatrix}.
\end{align*}
\]

The same procedure applied to governing equations (A.1a)-(A.1c). First, it yields at the order \( i / \varepsilon^{-1} \)

\[
\frac{d\Phi}{dt} \begin{bmatrix} u^{(0)}, b^{(0)}, \Theta^{(0)} \end{bmatrix} = -p^{(0)} k, 0, 0.
\]

The dot product of the first equation (A.6) with \( \nabla \Phi \), under constraint (A.5a), yields \( p^{(0)} = 0 \). Then, we obtain the Hamilton-Jacobi equation

\[
\frac{d\Phi}{dt} = 0.
\]

Finally, taking the spatial gradient of equation (A.7) reads the eikonal equation together with its initial condition (Lifschitz & Hameiri 1991)

\[
\frac{dk}{dr} = -\left( \nabla U_0 \right)^T k, \quad k(r, 0) = k_0, \quad |k(r, t)| = |k_0|.
\]
Now, by using (A.7) and (A.8), equations (A.1a)-(A.1c) yield at the next asymptotic order $\varepsilon^0$

$$-i k \left[ p^{(1)} + \alpha_B \tilde{B}_0 \cdot b^{(0)} \right] = \left( \frac{d}{dt} + \nabla U_0 + 2 \Omega \times \right) u^{(0)} - \alpha_T \Theta^{(0)} g - i \alpha_B (\tilde{B}_0 \cdot k) b^{(0)}, \quad \text{(A.9a)}$$

Equations (A.9b)-(A.9c) are transport equations for the magnetic field and the temperature amplitudes. Applying the dot product

$$\frac{d \hat{d}^{(0)} - i (\tilde{B}_0 \cdot k) u^{(0)} + (\nabla U_0) b^{(0)}, \quad \text{(A.9b)}$$

and with $\hat{D}/\hat{t}$ the Lagrangian derivative. Therefore, equations (A.13) are interpreted as ordinary differential equations along the fluid trajectories of the background flow $U_0$ for the amplitudes $(u^{(0)}, \Theta^{(0)}, \tilde{\xi}^{(0)})$. In addition, the initial conditions satisfy

$$u^{(0)}(0) \cdot k_0 = 0, \quad b^{(0)}(0) \cdot k_0 = 0, \quad \text{(A.14)}$$

such the solenoidal conditions for the velocity and the magnetic field hold at any time. Sufficient conditions for instability are obtained when (e.g. Lifschitz & Hameiri 1991; Lebovitz & Lifschitz 1992; Lifschitz & Lebovitz 1993)

$$\lim_{t \to \infty} \left( \left| u^{(0)} \right| + \left| b^{(0)} \right| + \left| \Theta^{(0)} \right| \right) = \infty \quad \text{(A.15)}$$

for given $[X_0, k_0]$ and with suitable initial conditions for $[u^{(0)}, b^{(0)}, \Theta^{(0)}]$.

**Appendix B: MAC modes in triaxial ellipsoids**

In this appendix, we present a new method to compute the three-dimensional hydromagnetic eigenmodes of stratified Boussinesq fluids contained within rigid triaxial ellipsoids. This approach relies on a fully global, explicit spectral method in ellipsoids, in which the velocity field is described by polynomial finite-dimensional Galerkin bases (Vidal & Cébron 2017). The algorithm has been implemented within the code SHINE (https://bitbucket.org/vidalje/shine). It has been benchmarked successfully for the Coriolis modes in ellipsoids (Vantieghem 2014; Lemasquerier et al. 2017), while the fast and slow hydro-magnetic solutions have been validated for the Malkus field in spheres (Malkus 1967; Zhang et al. 2003), spheroids (Kerswell 2014) and triaxial ellipsoids (Vidal et al. 2016).

**Appendix B.1: Assumptions**

We work in dimensional variables for the sake of generality, and use the notations introduced in the main text. We consider a diffusionless, incompressible electrically conducting fluid, contained within a triaxial ellipsoid of semi-axes $(a, b, c)$. The fluid is stratified under the gravity field $g^*$ in the Boussinesq approximation. The fluid is contained within an ellipsoidal container, which is rotating at the angular velocity $\Omega$ in the inertial frame. We expand the velocity, the temperature and the magnetic field as small perturbations $[u^*, \Theta^*, b^*](r, t)$ around an equilibrium state of rest $[0, T_0^*, B_0^*](r)$. In the linear approximation, the dimensional governing equations are

$$\frac{\partial u^*}{\partial t} = -2 \Omega \times u^* - \nabla p^* - \alpha_T \Theta^* g^* + \alpha_B \left[ (\nabla \times b^*) \times B_0^* + (\nabla \times B_0^*) \times b^* \right], \quad \text{(B.1a)}$$

$$\frac{\partial \Theta^*}{\partial t} = -(u^* \cdot \nabla) T_0^*, \quad \text{(B.1b)}$$

$$\nabla \cdot u^* = \nabla \cdot b^* = 0, \quad \text{(B.1c)}$$

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with \( \alpha_B = (\rho_M \mu_0)^{-1} \) and \( p^* \) the hydrodynamic pressure. By taking the time derivative of equations (B.1), we can obtain a single wave-like equation of second order in time for the velocity perturbation \( u^* \). This reads

\[
\frac{\partial^2 u^*}{\partial t^2} + 2\Omega \times \frac{\partial u^*}{\partial t} + \alpha_F (u^* \cdot \nabla) g^* + f_m^*,
\]

with the Lorentz force

\[
f_m^* = \alpha_B (\nabla \times B_0^*) \times (\nabla \times (u^* \times B_0^*)) + \alpha_B [\nabla \times (\nabla \times (u^* \times B_0^*))] \times B_0^*.
\]

Note that equations (B.1) cannot be recast into a single equation for the velocity perturbation \( u^* \) in the presence of an arbitrary basic flow \( B_0^* \), e.g. flow (7) considered in the main text. However, we point out that some well-chosen ellipsoidal equilibrium configurations could be reduced to a single equation (e.g. in spheres (Friedlander 1989)).

Finally, equation (B.2) is supplemented by the non-penetration boundary conditions

\[
u \cdot \mathbf{n}_e = 0, \quad B_0^* \cdot \mathbf{n}_e = 0.
\]

with \( \mathbf{n}_e \) the unit outward vector normal to the ellipsoidal boundary. We emphasise that alternative boundary conditions for the background magnetic field cannot be considered with the polynomial Galerkin description, at least to investigate consistently all the hydromagnetic modes. Allowing a non-zero normal magnetic field at the boundary would create a surface electrical density current, generating a Lorentz force \( f_m^* \) in the form of a discontinuous Dirac function distributed on the boundary (Friedlander & Vishik 1990). This would lead to spurious diffusionless solutions for the slow hydromagnetic modes. However, we would expect the fast hydromagnetic modes (i.e. Coriolis modes) to be only barely affected by the magnetic boundary condition, since the Lorentz force in momentum equation (B.2) has only second-order effects on the fast modes.

Appendix B.2: Galerkin method

We employ a Galerkin method to describe the velocity field. We seek a finite-dimensional Galerkin expansion of the modes as

\[
[u^*, p^*](r, t) = [\tilde{u}^*, \tilde{p}^*](r) \exp(i\omega_i t), \quad \tilde{u}^* = \sum_{i=1}^{\infty} \gamma_i \tilde{u}_i,
\]

where \( \omega_i \) is the angular frequency, \( \{\gamma_i\} \) modal complex coefficients and \( \{\tilde{u}_i(r)\} \) real-valued basis Galerkin elements (see later).

First, we rewrite equation (B.2) in the symbolic form

\[
(-\omega_i^2 + i\omega_i \mathcal{A}_1 + \mathcal{A}_0) \tilde{u}^* = i\omega_i \nabla \tilde{p}^*,
\]

where \( \{\mathcal{A}_1, \mathcal{A}_0\} \) are two linear operators. The basis elements \( \{\tilde{u}_i(r)\} \) are made of linear combinations of Cartesian monomials \( \{x^j y^k z^l\}_{l=1,j+k=0}^{\infty} \) satisfying

\[
\nabla \cdot \tilde{u}_i = 0, \quad \tilde{u}_i \cdot \mathbf{n}_e = 0.
\]

Expansion (B.5) is similar to expansions used in the finite-element method (FEM). However, compared to the traditional FEM, our basis elements \( \{\tilde{u}_i(r)\} \) are global polynomials, infinitely continuously differentiable in ellipsoids. The mathematical completeness of the polynomial expansion for incompressible fluids is then ensured by using the Weierstrass approximation theorem (Backus & Rietbout 2017; Ivers 2017). Hence, this method is a rigorous spectral method in ellipsoids. Several Cartesian expansions have been proposed, see a comparison in Vidal & Cébron (2017).

Then, we truncate the series (B.5) at a given polynomial degree \( n \) such that \( i + j + k \leq n \). In the absence of any stratified or magnetic effect, the Coriolis operator is exactly closed within the considered polynomial bases (e.g. Kerswell 1993b; Backus & Rietbout 2017). Thus, the Coriolis modes are exactly described by the polynomial description (see computations in Vantieghem 2014; Lemasquerier et al. 2017). Note that fast and slow MC modes also admit exact polynomial descriptions for background magnetic fields which are linear in the Cartesian space coordinates (Malkus 1967; Vidal et al. 2016). For any other practical configuration, we have to choose a maximum polynomial degree \( n \) to ensure a good enough convergence of the desired modes (since higher order bases are excited by the buoyancy and Lorentz forces). We substitute the truncated expansion into equation (B.6), yielding the quadratic eigenvalue problem

\[
(-\omega_i^2 A_2 + i\omega_i A_1 + A_0) \gamma = 0,
\]

where \( \gamma = (\gamma_1, \gamma_2, ..., \gamma_i)^T \) is the eigenvector and \( \{A_2, A_1, A_0\} \) are three real-valued matrices. Their elements are given by the Galerkin projections over the ellipsoidal domain

\[
A_{2,ij} = \int_V \tilde{u}_i^* \cdot \tilde{u}_j^* \, dV,
\]

\[
A_{1,ij} = \int_V \tilde{u}_i^* \cdot (\mathcal{A}_1 \tilde{u}_j^*) \, dV,
\]

\[
A_{0,ij} = \int_V \tilde{u}_i^* \cdot (\mathcal{A}_0 \tilde{u}_j^*) \, dV.
\]

The volume integral for a given Cartesian monomial \( x^j y^l z^k \) is exactly given by

\[
\int_V x^j y^l z^k \, dV = \frac{2\alpha_i^{j+l+1} \Gamma\left(\frac{j+1}{2}\right) \Gamma\left(\frac{k+1}{2}\right)}{3 + i + j + k} \times \beta\left(\frac{i + j + k + 1}{2}\right).
\]

if \( i, j, k \) are all even and vanishes otherwise, where \( \beta(i, j) \) is the transcendental beta function defined as a function of the Gamma function \( \Gamma(i) \) by

\[
\beta(i, j) = \frac{\Gamma(i) \Gamma(j)}{\Gamma(i + j)} = \frac{(2)!}{2^{i+j} \sqrt{\pi}}.
\]

Note that the pressure term vanishes in equation (B.8) by virtue of the divergence theorem, such that an explicit decomposition for the pressure is not required.

Appendix B.3: Hydromagnetic modes

We show in figure B.1 the dimensionless eigenfrequency \( \omega_i \) of MAC modes in full ellipsoids, for the relevant weak field regime \( Le \leq 10^{-1} \). We have considered an arbitrary reference configuration to illustrate several representative properties of the modes. We identify three families of waves in neutrally stratified fluids (figure B.1a), in agreement with investigations in spherical geometries (e.g. Schmitt 2010; Labbé et al. 2015). First, the high frequency branch represents fast Magneto-Coriolis (MC)
modes (Malkus 1967; Labbé et al. 2015). They are similar to pure Coriolis (or inertial) modes (Greenspan 1968; Vantieghem 2014; Backus & Rieutord 2017), with a dimensionless spectrum bounded by $|\omega_i| \leq 2$ in the weak field regime $Le \ll 1$. These modes are regular in space and only weakly affected by large-scale magnetic fields in weakly deformed spheres (e.g. Schmitt 2010; Labbé et al. 2015), as observed by the frequency dependence on $Le$ in the figure. Note that they behave differently than the singular modes localised on attractors (e.g. Rieutord & Valdettaro 1997, 2018), that only exist in shells because the mathematical problem is ill-posed (Rieutord et al. 2000). Second, the low frequency branch represents slow Magneto-Coriolis modes. Their typical (dimensionless) frequency scales as $|\omega_i| \propto Le^2$. In addition, the third intermediate branch represents toroidal Alfvén modes (Labbé et al. 2015), scaling as $|\omega_i| \propto Le$. They are usually filtered out in reduced models, e.g. in local models considering uniform fields. They exist when the current direction $\nabla \times B_0$ of the basic state is misaligned with the spin rotation axis.

Then, we show the spectrum of MAC modes in stratified fluids in figure B.1b. The aforementioned hydromagnetic modes still exist in stably stratified interiors, yielding fast and slow MAC waves. However, their properties in the presence of buoyancy and magnetic fields are rather complex in spherical-like domains (Friedlander 1987). On the one hand, fast MAC modes and gravito-inertial modes are barely modified by magnetic fields, as illustrated in figure B.1b when $Le \ll 1$. However, they strongly depend on the stratification (Friedlander & Siegmann 1982b). On the other hand, slow MC modes can be strongly affected by the magnetic field and the stratification (Friedlander 1987). Finally, the buoyancy force also sustains high frequency internal gravity modes. They can be affected by the rotation, yielding gravito-inertial modes (Friedlander & Siegmann 1982b).

**Appendix C: Mixed resonances of MAC waves**

In this appendix, we investigate the nonlinear couplings of hydrodynamics waves for the tidal instability. We use the same dimensionless variables as in the main text. Resonance condition (12) can only be satisfied if the tidal instability involves fast MAC waves (i.e. inertial or gravito-inertial waves) coupled with either fast or Magneto-Coriolis (slow MC) waves (Kerswell 1993a, 1994). Indeed, in the astrophysical regime $Le \ll 1$, the illustrative spectrum in figure B.1 clearly shows that no triadic couplings are effective in ellipsoids between two slow MC waves. Thus, the couplings of slow MC waves with the equilibrium tide flow cannot be advocated in stellar interiors.

Second, the mixed couplings between slow and fast hydromagnetic waves is not forbidden in diffusionless fluids. In the weak field regime $Le \ll 1$, Kerswell (1993a, 1994) obtained that the typical diffusionless growth rate of the tidal instability involving mixed couplings scales as (in dimensionless form)

$$\sigma \propto Le^4 \beta_0.$$  \hspace{1cm} (C.1)

However, this diffusionless growth rate must be larger than the (laminar) Joule damping rate of the slow MC waves, i.e. $\tau_{\text{j}} \propto -Em |k_0|^2$ in the local theory (Rincon & Rieutord 2003; Sreenivasan & Narasimhan 2017). This gives the typical upper bound on the wave vector

$$|k_0|^2 \lesssim \frac{Le^4}{Em} \beta_0.$$ \hspace{1cm} (C.2)

In short-period binaries, the typical value for the equatorial ellipticity is $\beta_0 \sim 10^{-3} - 10^{-2}$ (see table 2). As given in table 1, we have also the typical numbers $Em \leq 10^{-10}$ and $Le \leq 10^{-4}$. Then, condition (C.2) yields the upper bound $|k_0| \ll 1$. This is incompatible with the short-wavelength stability theory, which requires $|k_0| \gg 1$. Physically, this shows that the (laminar) Joule damping rate is always larger than the diffusionless growth rate in non-ideal fluids, for any resonance involving slow MC waves in the regime $Le \ll 1$. Therefore, mixed couplings of fast/slow waves can be discarded for the tidal instability in realistic stellar interiors.
Appendix D: Weakly eccentric synchronised orbits

Appendix D.1: Libration forcing

In this appendix, we consider synchronous stratified binary systems moving on weakly eccentric coplanar orbits. Note that the following results are also relevant for stratified moons or gaseous planets orbiting around a massive central body (e.g. Kerswell & Malkus 1998; Cébron et al. 2012b; Lemasquerier et al. 2017). We consider a diffusionless tidal model of the tidally deformed fluid body, characterised by an equatorial ellipticity $\beta_0$. The fluid body is rotating at the uniform angular velocity $\Omega_{s}$, aligned in the inertial frame with the orbital angular velocity of the companion along $\mathbf{I}_z$. We use the dimensionless variables introduced in §2, i.e. taking $(\Omega_{s})^{-1}$ as the relevant time scale. Due to the weak orbital eccentricity $e \ll 1$, the orbital angular velocity has periodic time variations. For the sake of generality, we assume that the tidal forcing has the following (dimensionless) expression, at the leading order in the eccentricity

$$\Omega(t) = 1 + \epsilon_t \cos (ft).$$

where $f$ is the dimensionless frequency of the forcing and $\epsilon_t \ll 2e$ the dimensionless amplitude. Forcing (D.1) is known as longitudinal librations. For this tidal forcing, the equilibrium tide velocity field has the following form in the central frame

$$U_0(r, t) = -\epsilon_t \cos(ft) \left[ -(1 + \beta_0) \mathbf{I}_x + (1 - \beta_0) x \mathbf{I}_y \right].$$

The above tidal flow (D.2) is prone to the libration-driven elliptical instability (LDEI), which is quite similar to the tidal instability in non-synchronised systems (e.g. Kerswell & Malkus 1998; Cébron et al. 2012b; Vidal & Cébron 2017).

Appendix D.2: Resonance condition of the LDEI

LDEI is a fluid instability due to sub-harmonic resonances between two waves of angular frequency $|\omega_i|$ interacting with the basic flow (D.2). By analogy with formula (13) in non-synchronised systems, the sub-harmonic resonant condition yields

$$|\omega_i| = f/4.$$  \hspace{1cm} (D.3)

The four kinds of waves $[H_1, H_2, E_1, E_2]$, introduced §3.2, can be nonlinearly coupled in the instability mechanism. We show the nature of the waves satisfying condition (D.3) in figure D.1.

Fig. D.1: Waves at the sub-harmonic resonance condition (D.3) for synchronised systems, as a function of the (dimensionless) forcing frequency $f$ and $N_0/\Omega_{s}$. The other notations are identical to the ones introduced in the main text. White regions: no compatible waves satisfying (D.3). Stars (yellow area): hyperbolic waves $H_1$, Right slash (purple area): hyperbolic waves $H_2$, Dots (green area): elliptic waves $E_1$. Back slash (blue area): elliptic waves $E_2$. The classical allowable region of the instability is $0 \leq f \leq 4$ in neutral fluids.

The classical allowable range of the instability is $0 \leq f \leq 4$ (e.g. Cébron et al. 2012c), in which only triadic couplings of inertia-gravity waves $[H_1, H_2, E_1, E_2]$ are involved. In this frequency range, the instability is trapped along critical latitudes for a strong enough stratification, i.e. when $N_0/\Omega_{s} \gg 1$. Similar to the non-synchronised configurations, it turns out that the largest growth rate is unaffected by the ratio $N_0/\Omega_{s}$ on these critical latitude. Thus, they are predicted by the diffusionless formula obtained in neutral fluids (see formula 4 of Cébron et al. 2012c).

In the other frequency range $f > 4$, the instability is only due to triadic couplings of internal-gravity waves $[E_1, E_2]$ modified by the rotation. Moreover, the instability only exists for strong enough stratification $N_0/\Omega_{s} \gg 1$.

Appendix D.3: Asymptotic growth rates of the LDEI

As in sections 3.2.3 and 3.2.4, the local stability analysis provides analytical expressions of the diffusionless growth rates in the equatorial plane and on the rotation (polar) axis. In the equatorial plane, the resonance condition (D.3) becomes

$$\sqrt{4 + N_0^2 \frac{x_0^2}{\beta_0} \cos \theta_0} = \pm \frac{f}{2},$$

whereas on the rotation axis we have

$$\sqrt{4 \cos^2 \theta_0 + N_0^2 \frac{x_0^2}{\beta_0} \sin^2 \theta_0} = \pm \frac{f}{2}.$$  \hspace{1cm} (D.5)

Then, the diffusionless growth rate in the equatorial plane is

$$\sigma = \left(1 + \frac{f^2}{16}\right) \frac{|\beta_0 - N_0^2 \frac{x_0^2}{\beta_0} (\beta_0 - \beta_1)|}{4 + N_0^2 \frac{x_0^2}{\beta_0}} \epsilon_t$$

for a general baroclinic background state $\beta_0 \neq \beta_1$. On the rotation axis, the diffusionless growth rate is given by

$$\sigma = \frac{(16 + f^2)(1-4N_0^2 \frac{x_0^2}{\beta_0} f^{-2}) \beta_0 \epsilon_t}{16(4 - N_0^2 \frac{x_0^2}{\beta_0})}.$$  \hspace{1cm} (D.7)

Naturally, we recover equation (4) of Cébron et al. (2012c), obtained for neutral fluids ($N_0 = 0$). Note that equation (25) of Cébron et al. (2013), obtained in the equatorial plane for a buoyancy force of the order $\beta_0$, is not recovered by equation (D.6). Indeed, it turns out that their equation (25) is erroneous since they artificially set $\theta_0$ to its hydrodynamic value $2 \cos \theta_0 = \pm f/2$, instead of using its correct value given by equation (D.4). Finally, by analogy with the arguments given in the main text for non-synchronised systems, the largest diffusionless growth rate in the stellar interior will be insensitive to the strength of the stratification, yielding the value for neutral fluids (Cébron et al. 2012c, 2013; Vidal & Cébron 2017) recovered in formula (D.6) for $N_0 = 0$.

Note finally that formula (30b) also provides exactly the Joule damping rate of the LDEI in neutral fluids ($N_0 = 0$). Besides,
formulas of Cébron et al. (2012a,b) are recovered in the limit $|k_0| \gg 1$ by using the LDEI resonance condition to set $\theta_0$, i.e. $\cos \theta_0 = \pm f/4$ for $N_0 = 0$.

**Appendix D.4: Mixing-length theory**

We can build a mixing-length theory to get a phenomenological prescription of the turbulent mixing in weakly eccentric synchronised orbits, by analogy with non-synchronised orbits. The main difference with non-synchronised systems is that the typical turbulent velocity $u_t$ should scale as (Favier et al. 2015; Grannan et al. 2016)

$$u_t \propto \alpha_{1} \epsilon \beta_{0} \Omega_s$$  \hspace{1cm} (D.8)

yielding the prescription

$$\tau_t \propto \frac{K_\alpha}{\epsilon^2 \beta_0^2 \Omega_s}$$  \hspace{1cm} (D.9)

with $K_\alpha \sim 30 – 50$, see the discussion of formulas (41). Hence, the time scale for the turbulent Ohmic diffusion of the fossil field ought to be reduced in synchronised systems (compared to non-synchronised ones) by using formula (D.9).